

Permutations Realized by Dynamical Systems

Kate Moore

January 15, 2018

Overview

Example: $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 4x(1 - x)$, initial condition $x_0 = .10$.

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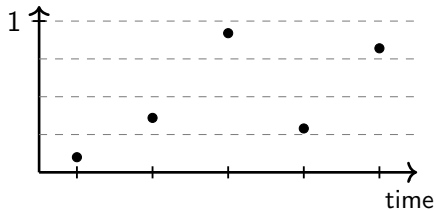
$$x_0 = .10$$

$$f(x_0) = .36$$

$$f(f(x_0)) = .92$$

$$f^3(x_0) = .29$$

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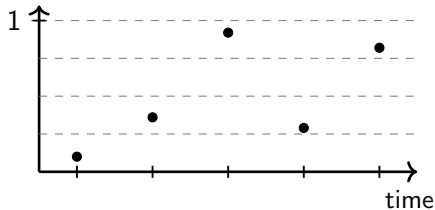
1

3

5

2

4



The pattern of length 5 for f at $x_0 = .10$ is $\pi = \mathbf{13524}$.

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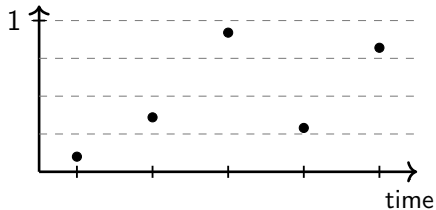
1

3

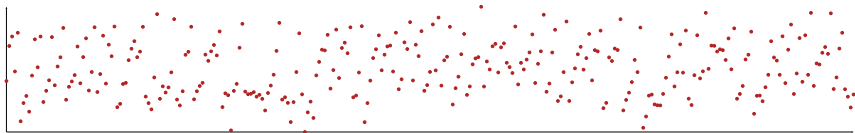
5

2

4



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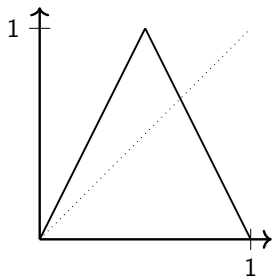


Motivation: Understand time series in the context of dynamical systems.

Allowed Patterns

What are the allowed patterns?

Example: Let $T(x) = \min\{2x, 2(1-x)\}$.

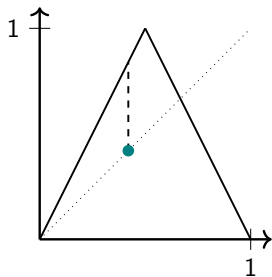


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Example: Let $T(x) = \min\{2x, 2(1-x)\}$.

$(x, T(x), T^2(x)) = (.42, .84, .32) \rightsquigarrow 231$

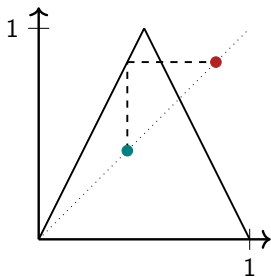


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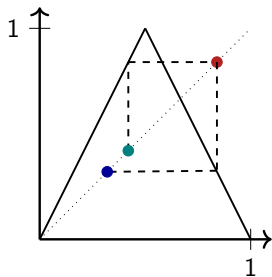


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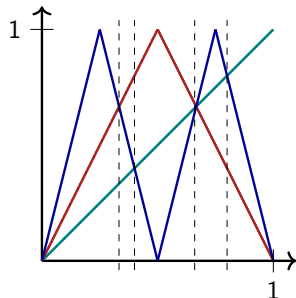
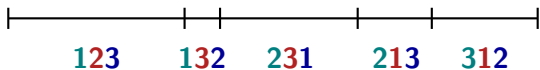


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n	3	4	5	6	7	8	10	12
$ \text{Allow}_n(T) $	5	12	31	75	178	414	2137	10,525

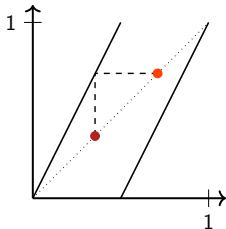
Best-known Bounds: (Elizalde & M.)

Notice: 321 is forbidden \rightsquigarrow 3214, 4213, 126534... are forbidden.

Permutation Structure of Periodic Points

Example: $F_2(x) = 2x \pmod{1}$. The point $x_0 = \frac{11}{31}$ is 5-periodic.

$$\left(\frac{11}{31}, \frac{22}{31}, \frac{13}{31}, \frac{26}{31}, \frac{21}{31} \right)^\infty$$

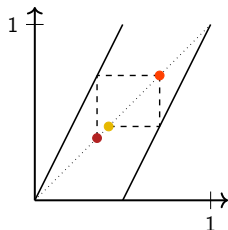


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$$\left(\frac{11}{31}, \frac{22}{31}, \frac{13}{31}, \frac{26}{31}, \frac{21}{31} \right)^\infty \longrightarrow \pi = \mathbf{14253}$$

Other representatives of the periodic orbit give cyclic rotations of π .

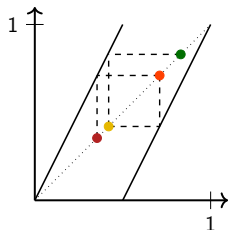


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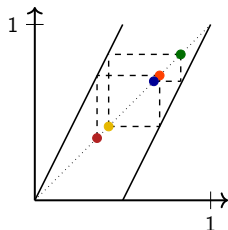


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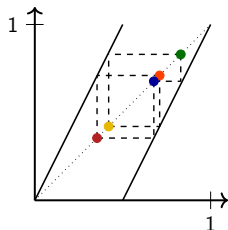


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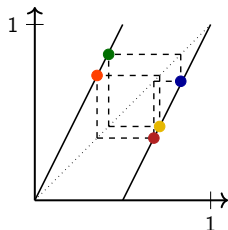


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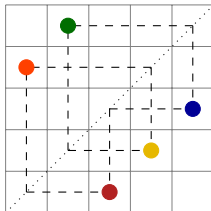
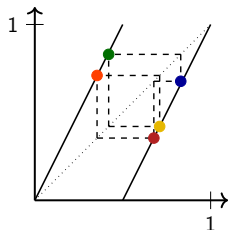


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Other representatives of the periodic orbit give cyclic rotations of π .



$$\hat{\pi} = (1, 4, 2, 5, 3) = 45123 \in \mathcal{C}_5$$

Key Idea: Cycles $\hat{\pi}$ obtained in this way have at most one descent.

Permutation Structure of Periodic Points

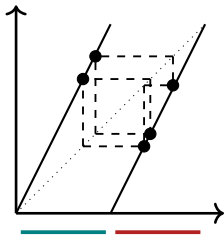
Let $F_2(x) = 2x \bmod 1$ and $I_0 = [0, \frac{1}{2})$, $I_1 = [\frac{1}{2}, 1)$.

The itinerary for the 5-periodic orbit of $\frac{11}{31}$ is:

$$\left(\frac{11}{31}, \frac{22}{31}, \frac{13}{31}, \frac{26}{31}, \frac{21}{31} \right)^\infty \rightarrow (\mathbf{0} \mathbf{1} \mathbf{0} \mathbf{1} \mathbf{1})^\infty$$

Which is the binary expansion:

$$\frac{11}{31} = \frac{\mathbf{0}}{2} + \frac{\mathbf{1}}{2^2} + \frac{\mathbf{0}}{2^3} + \frac{\mathbf{1}}{2^4} + \frac{\mathbf{1}}{2^5} + \dots$$



Permutation Structure of Periodic Points

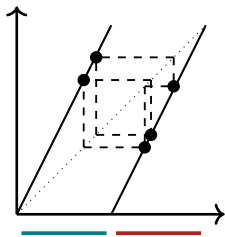
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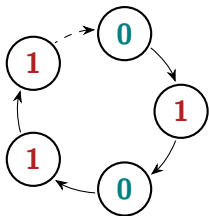
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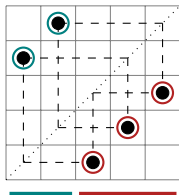
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(Gessel & Reutenauer, '93)



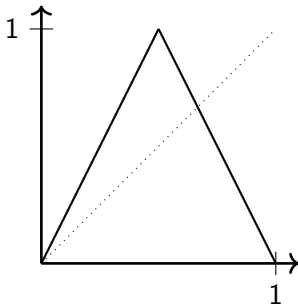
Binary necklaces with
distinct rotations of length n



Cycles of length n with
(at most) one descent

The Shape of Patterns

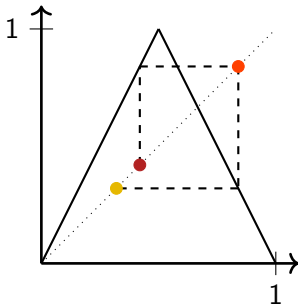
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$$\pi = 261453$$

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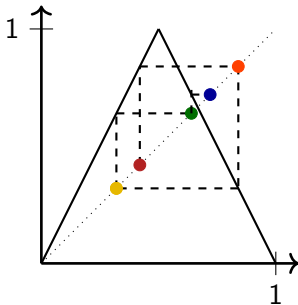
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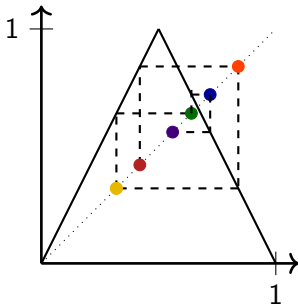
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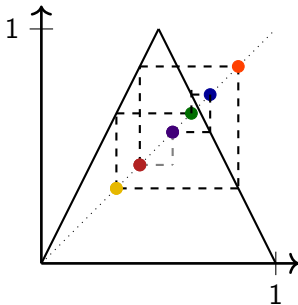
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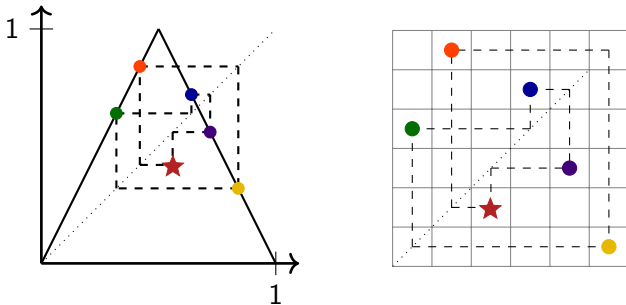
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$$\pi = 261453 \longrightarrow \hat{\pi}^* = (\star, 6, 1, 4, 5, 3) = 46\star 531$$

Key Idea: If $\pi \in \text{Allow}_n(T)$, then $\hat{\pi}^*$ is unimodal.

We ignore the value covered by \star .

(Except in the case that it is the first or last digit of $\hat{\pi}^*$.)

Signed Shifts

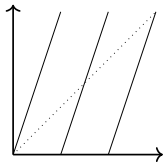
The transformation:

$$\pi = \pi_1 \pi_2 \dots \pi_n \longrightarrow \hat{\pi}^* = (\star, \pi_2, \dots, \pi_n) = \hat{\pi}_1^* \hat{\pi}_2^* \dots \hat{\pi}_n^*$$

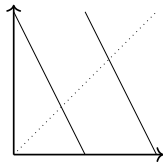
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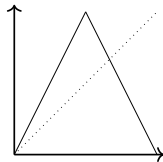
Theorem (Elizalde & M.): $\pi \in \text{Allow}(\Sigma_\sigma)$ if and only if $\hat{\pi}^*$ has the same monotonicity as σ and π is not “ σ -collapsed.”



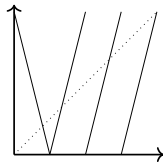
$\sigma = +++$
 $F_3(x) = 3x \pmod{1}$



$\sigma = --$
 $G_2(x) = -2x \pmod{1}$



$\sigma = +-$
Tent Map



$\sigma = -+++$

Topological Entropy and Patterns

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has finitely many monotone segments.

The *topological entropy* of f is

$$h^{\text{top}}(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \log(c_n),$$

where c_n is the number of monotone segments of f^n .

Example: $T(x) = \min\{2x, 2(1-x)\}$

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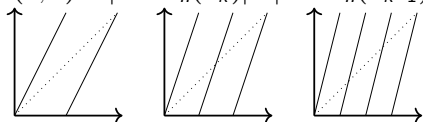
Theorem: (Bandt, Keller & Pompe '02)

$$h^{\text{top}}(f) = \lim_{n \rightarrow \infty} \frac{1}{n-1} \log(|\text{Allow}_n(f)|).$$

Important Idea: We do not require knowledge of f to estimate the topological entropy using patterns.

Enumerating Allowed Patterns

$$b(n, k) := |\text{Allow}_n(F_k)| \setminus |\text{Allow}_n(F_{k-1})|$$



k	2	3	4	5
$b(6, k)$	126	402	186	6

Theorem (Elizalde):

$$\sum_{k=2}^n b(n, k)x^k = (1-x)^n \sum_{k \geq 2} p(n, k)x^k,$$

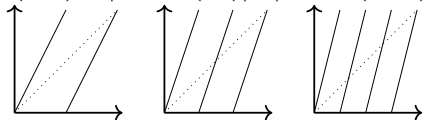
where

$$p(n, k) = \sum_{i=1}^{n-1} k^{n-i-1} \psi_k(i) + (k-2)k^{n-2},$$

and $\psi_k(i)$ is the number of k -ary primitive words of length i .

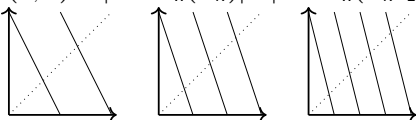
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$$\bar{b}(n, k) := |\text{Allow}_n(G_k)| \setminus |\text{Allow}_n(G_{k-1})|$$



k	2	3	4	5
$\bar{b}(6, k)$	140	408	168	4

Theorem (Elizalde & M.):

$$\sum_{k=2}^n \bar{b}(n, k) x^k = (1-x)^n \sum_{k \geq 2} \bar{p}(n, k) x^k.$$

where $\bar{p}(n, k)$ is equal to

$$\sum_{i=1}^{n-1} k^{n-i-1} \psi_k(i) + (k^2 - 2)k^{n-3} - 2 \sum_{j=1}^{k-1} j^{n-3} - 2 \sum_{\substack{c=1 \\ \text{odd}}}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^{k-1} \binom{c+k-j-2}{k-j} j^{n-2c-1} \psi_j(c),$$

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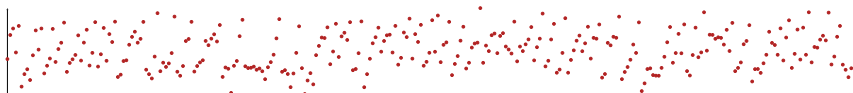
Enumerating Allowed Patterns (cont.)

$n \setminus k$	2	3	4	5	6	7
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

$n \setminus k$	2	3	4	5	6	7
3	6					
4	20	4				
5	54	62	4			
6	140	408	168	4		
7	336	2084	2208	408	4	
8	800	9152	19580	9820	964	4

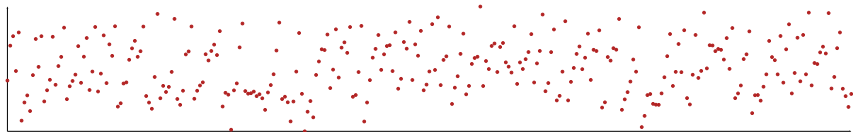
Values of $b(n, k)$ (left) and $\bar{b}(n, k)$ (right).

Corollary: The smallest forbidden patterns of $F_k(x) = kx \pmod{1}$ and $G_k(x) = -kx \pmod{1}$ are of length $k + 2$.



Lower Bounds on Entropy Using Patterns

Proposed Goal: Use patterns to obtain lower bounds on entropy in certain settings (e.g. shifts, unimodal, continuous, ...).

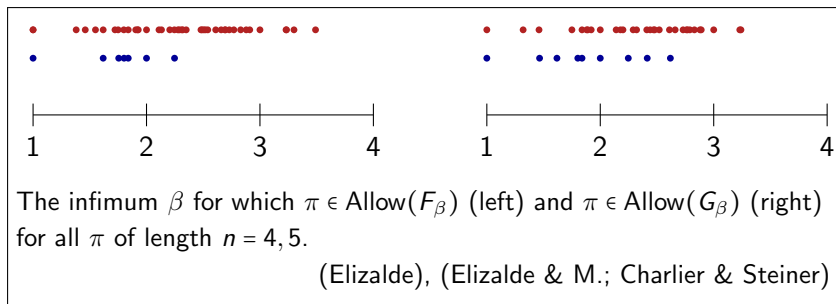


Relations with: Lower bound on entropy for any continuous map containing a periodic point with the given cycle structure, $\hat{\pi}$.

(Sarkovskii, Baldwin, and many others in the 80s).

Lower Bounds on Entropy Using Patterns

For $\beta > 1$, consider $F_\beta(x) = \beta x \bmod 1$ and $G_\beta(x) = -\beta x \bmod 1$.



Example: If we suppose our time series is generated by G_β and we observe $\pi = 15237864$, then we must have had

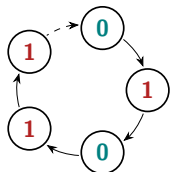
$$\beta \geq 3.154,$$

the largest real root of

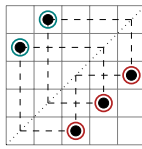
$$P_\pi(x) = x^4 - 4x^3 + 3x^2 - 2x + 3.$$

Thank You

Dynamical interpretation of combinatorial problems.



Binary necklaces with distinct rotations of length n



Cycles of length n with (at most) one descent

Permutation-based techniques for estimating entropy of time series.

