Patterns in Time Series and Dynamical Systems

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with Daryl DeFord
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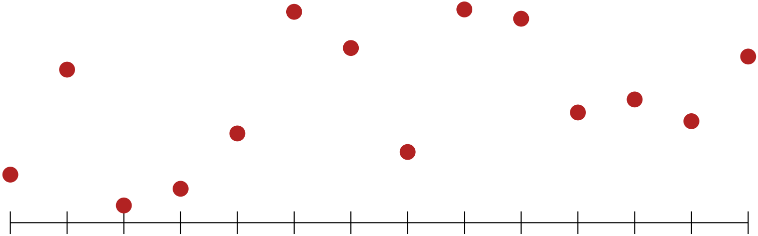
Dartmouth College

September 12, 2017

Patterns in Time Series

How unpredictable is our time series?



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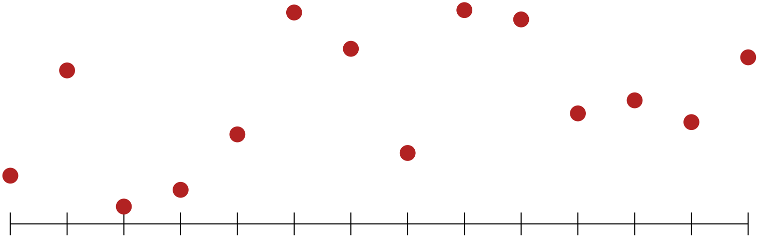

$$\underbrace{(0.21,0.68,0.07,0.15)}_{3412}$$

Patterns: $\{3412\}$

Patterns in Time Series

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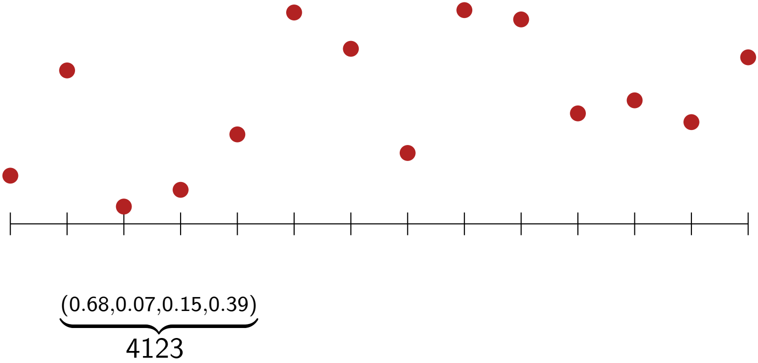

Patterns: $\{3412, 4123\}$

Patterns in Time Series

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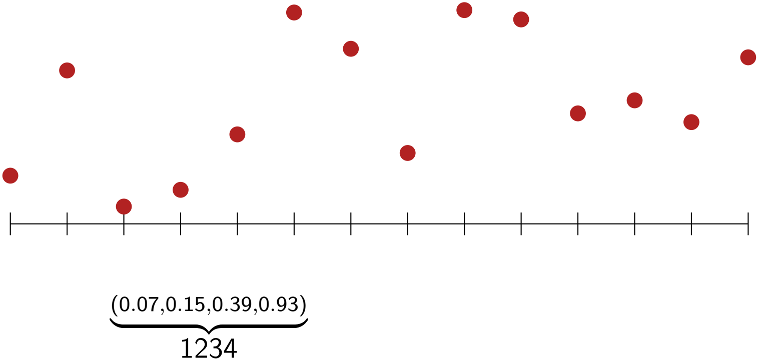
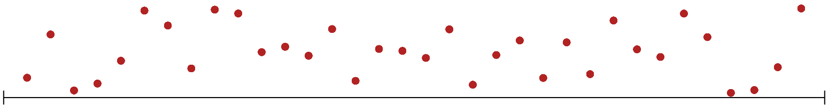


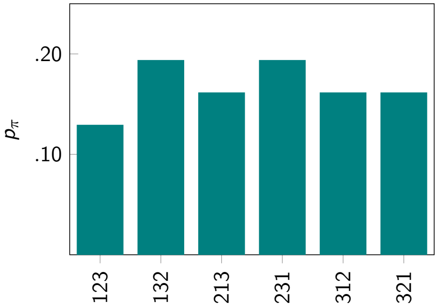
Patterns:

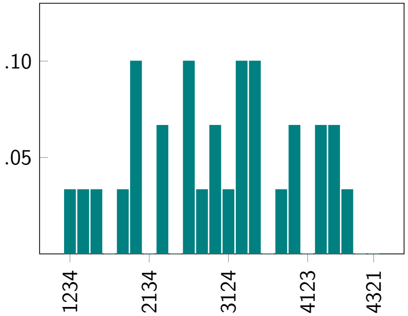
$\{3412, 4123, 1234, 1243, 2431, 4213, 2143, 1432, 4312, 4231, 2314\}$

Patterns in Time Series



The distribution of patterns of length $n = 3$ and $n = 4$.

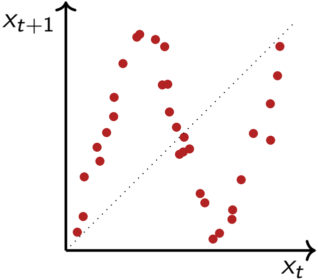


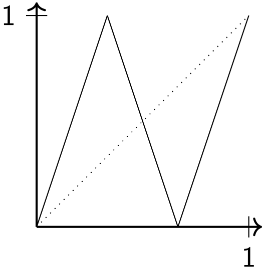


Time Series as Dynamical Systems

We can imagine our system as being defined by

$$x_{t+1} = f(x_t) + \epsilon.$$

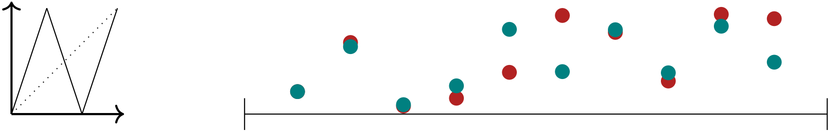




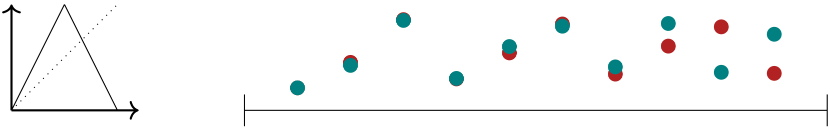
Question: With noise, ϵ, how quickly will predictions fail?

Time Series as Dynamical Systems

We measure the unpredictability of a time series according to how quickly noise propagates in the associated dynamical system.



$\{f^t(\mathbf{x_0})\}$ and $\{f^t(\mathbf{x_0})\}$ for $|\mathbf{x_0} - \mathbf{x_0}| = 0.001$

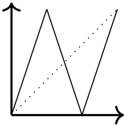


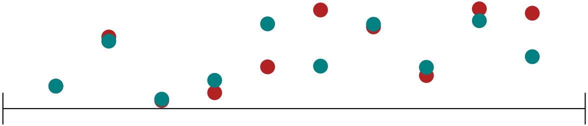
This sensitivity to the initial condition is measured by *entropy*.

Entropy of a Dynamical System

Goal: Develop meaningful techniques for estimating entropy from time series data.

- Topological entropy: The exponential growth rate of the number of distinguishable orbits at time n.





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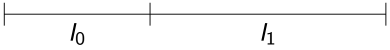
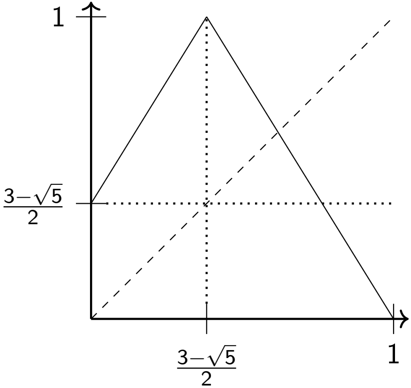
- Kolmogorov-Sinai entropy.

Topological Entropy

Suppose that f is piecewise monotone. Let c_n be the number of monotone segments of f^n, then

$$\text{TopEnt}(f) = \lim_{n\to\infty} \frac{1}{n} \log(c_n).$$





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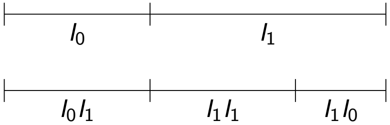

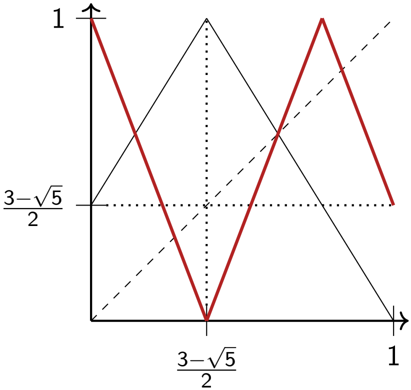

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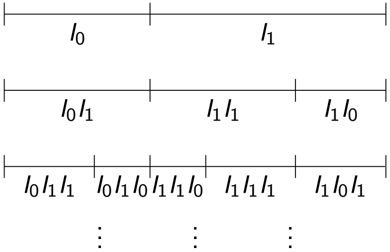

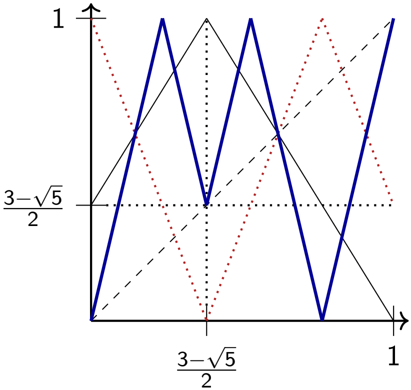

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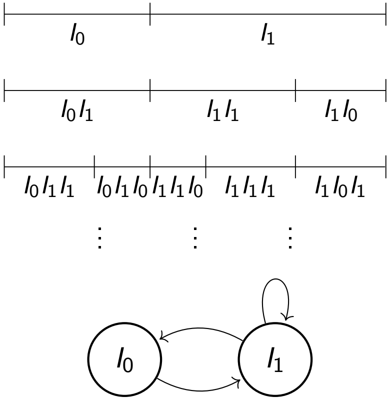

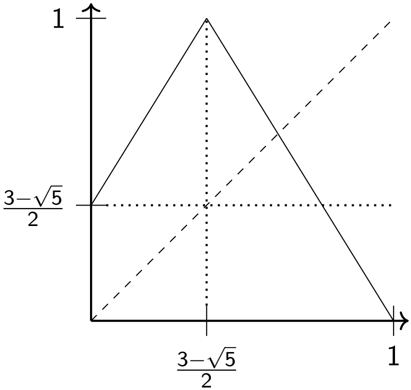

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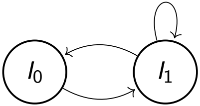
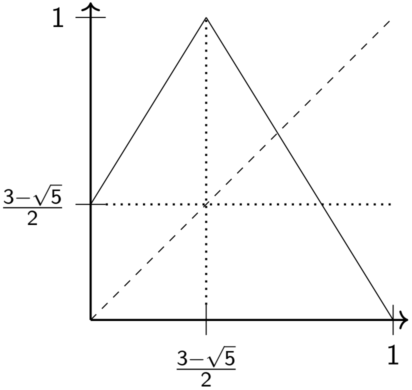


$$\text{Adj}(f) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \to \lambda_1 = \frac{1+\sqrt{5}}{2}$$

and so

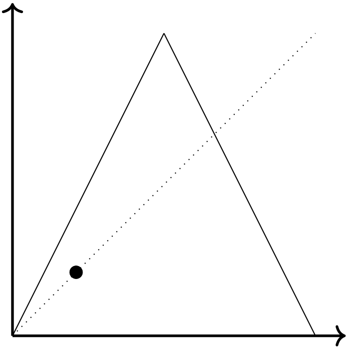
$$\text{TopEnt}(f) = \log(\frac{1+\sqrt{5}}{2})$$



Allowed Patterns

We say π is an *allowed pattern* of f if there exists an x such that

$$\mathrm{st}(x, f(x), f(f(x)), f^3(x), \ldots, f^{n-1}(x)) = \pi$$



$$(x, f(x), f^2(x), f^3(x), f^4(x))$$
$$= (.21, -, -, -, -)$$

We denote the set of allowed patterns of length n by $\mathrm{Allow}_n(f)$.

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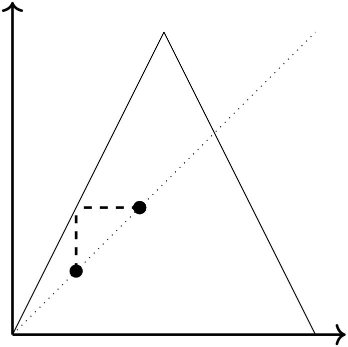

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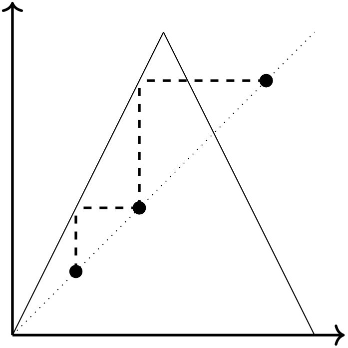

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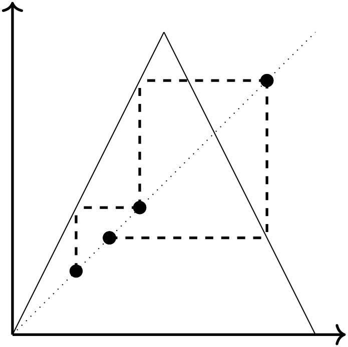

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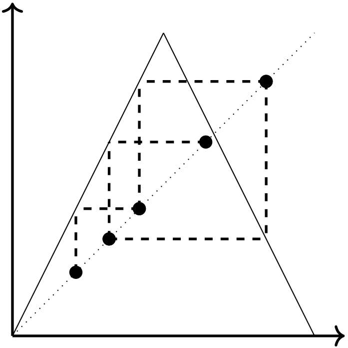
$$= \text{st}(.21, .42, .84, .32, .64) = 13524$$



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Topological Entropy and Patterns

Theorem: (Bandt, Keller and Pompe) If f is a piecewise monotone map of the interval,

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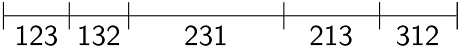

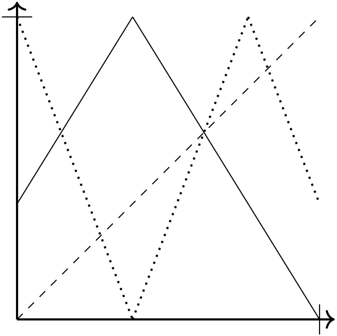

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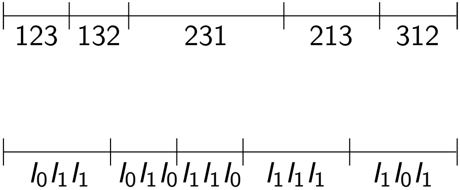

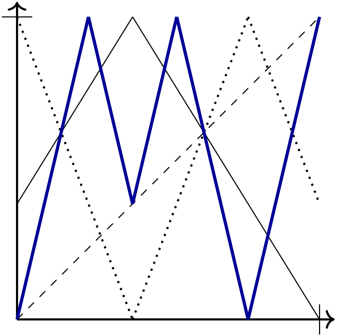

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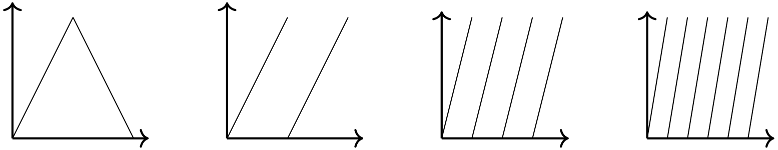

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=\infty$
$\lvert\mathrm{Allow}_n(\mathrm{Tent})\rvert$	5	12	31	75	178	414	$\sim 2^{n-2}$
$\lvert\mathrm{Allow}_n(\text{2-shift})\rvert$	6	18	48	126	306	738	$\sim 2^{n-1}$
$\lvert\mathrm{Allow}_n(\text{4-shift})\rvert$	6	24	120	714	4566	29004	$\sim c \cdot 4^{n-1}$
$\lvert\mathrm{Allow}_n(\text{6-shift})\rvert$	6	24	120	720	5040	40314	$\sim c \cdot 6^{n-1}$

Counting Allowed Patterns

Questions:

- In general, how accurate are the estimates of topological entropy?
- What is the length of the shortest forbidden pattern? How does this length relate to the entropy of the system?

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Theorem (Elizalde; Elizalde and M.): We obtained closed formulas for

$$|\text{Allow}_n(k\text{-shift})| \text{ and } |\text{Allow}_n(-k\text{-shift})|,$$

and reasonable bounds on $|\text{Allow}_n(\text{Tent})|$.

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and reasonable bounds on $|\text{Allow}_n(\text{Tent})|$.

Corollary: The smallest forbidden patterns of the k-shift have length $k+2$ and there are always exactly 6 of them. For example,

$$615243, 324156, 342516, 162534, 453621, 435261.$$

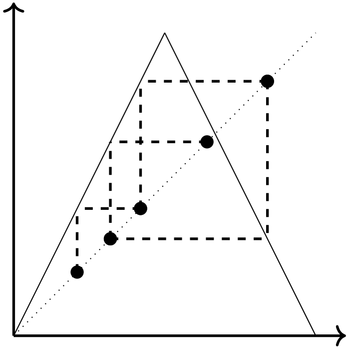
The smallest forbidden patterns of the $-k$-shift have length $k+2$ and there are always exactly 4 of them. For example,

$$123456, 654321, 123465, 654312.$$

The Shape of Patterns

The shape of patterns also conveys information.

$$\pi = 13524$$



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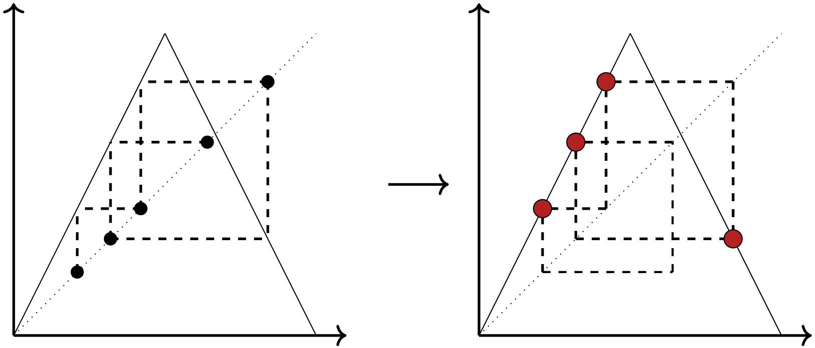
$$\pi = 13524 \longrightarrow \hat{\pi} = (\star, 3, 5, 2, 4) = 345\star 2$$



Example: If $\pi \in \text{Allow(Tent)}$, then $\hat{\pi}$ is unimodal.

Infimum Entropy of Patterns

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Question: Given a pattern π, what is the infimum topological entropy of a function f realizing π?

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Theorem (Elizalde; Elizalde and M.): We answer this question for the families of functions

$$F_\beta(x) = \beta x \mod 1 \quad \text{and} \quad G_\beta(x) = -\beta x \mod 1,$$

noting that $\text{TopEnt}(F_\beta) = \log(\beta)$ and $\text{TopEnt}(G_\beta) = \log(\beta)$.

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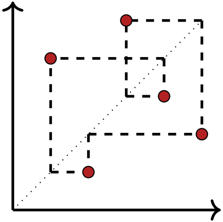
Question: What about other families of functions?

Some progress for symmetric tent maps (Archer and LaLonde).

Bounding Topological Entropy

For **continuous** functions, the patterns for periodic points give tight lower bounds on topological entropy (Baldwin and others).

Suppose a **continuous** function has a 5-periodic point with pattern $\pi = 52143$ (equiv. $21435, 14352, 43521, 35214$).



Minimal entropy continuous map with a periodic point of shape

$$\hat{\pi} = (5, 2, 1, 4, 3) = 41532$$

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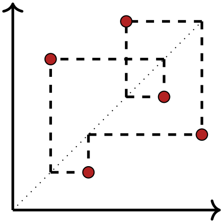

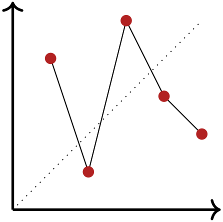

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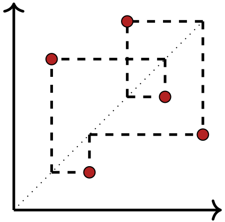

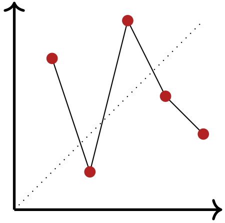

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 = 2.505$$

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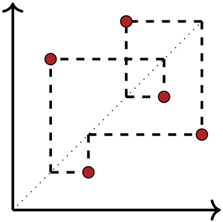

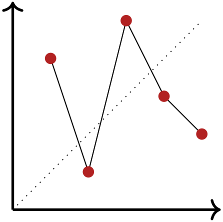

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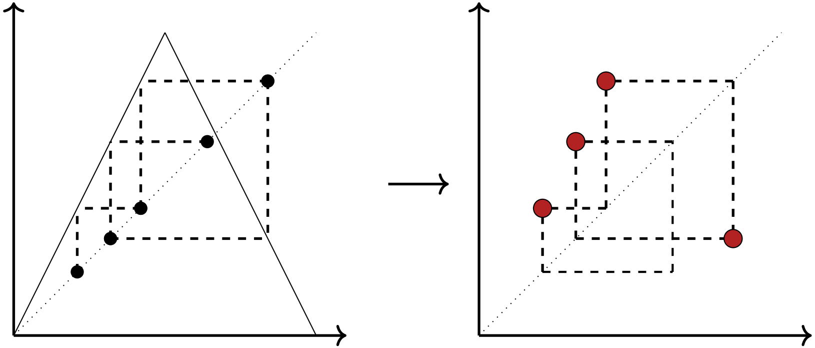
Minimal entropy continuous map with a periodic point of shape

$$\hat{\pi} = (5, 2, 1, 4, 3) = 41532 \text{ is } L_{\hat{\pi}} \text{ and } \mathrm{TopEnt}(L_{\hat{\pi}}) = \log(2.505).$$

Bounding Topological Entropy for Continuous Maps

Question: What is the infimum topological entropy of a **continuous** map realizing π? (not necessarily at a periodic point)

$$\pi = 13524 \longrightarrow \hat{\pi} = (\star, 3, 5, 2, 4) = 345\star2$$



Answer: Not yet.

Entropy of a Dynamical System

Goal: Develop meaningful techniques for estimating entropy from time series data.

- Kolmogorov-Sinai entropy: The system will eventually settle, spending a certain proportion of time in each sub-interval, how much unpredictability/surprise remains?

In English:	Random Alphabet:
$\mathbb{P}(Q \to U) \approx .987$,	$\mathbb{P}(A \to B) = 1/26 \approx .038$,
$\mathbb{P}(V \to E) \approx .686$.	$\mathbb{P}(A \to C) = 1/26 \approx .038$.

Kolmogorov-Sinai Entropy

Let μ be an invariant probability measure of a piecewise monotone function $f : I \to I$ (i.e. $\mu(f^{-1}(A)) = \mu(A)$ for measurable sets A).

Definition: Let $\mathcal{P}_n$ be the partition of I into intervals where f^n is monotone. Then

$$h_\mu := \lim_{n\to\infty} \frac{1}{n} \sum_{I_P \in \mathcal{P}_n} -\mu(I_P) \log(\mu(I_P)).$$

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Theorem (Bandt, Keller and Pompe): Let $\mathcal{A}_n$ be the partition of I into sets of the form

$$I_\pi = \{x : \mathrm{st}(x, f(x), \ldots, f^{n-1}(x)) = \pi\}.$$

Then

$$h_\mu = \lim_{n\to\infty} \frac{1}{n-1} \sum_{I_\pi \in \mathcal{A}_n} -\mu(I_\pi) \log(\mu(I_\pi)).$$

Permutation Entropy

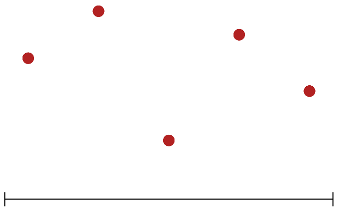
Define the permutation entropy of a time series by

$$\overline{\mathrm{PE}}_n = \frac{-1}{n-1} \sum_{\pi \in \mathcal{S}_n} p_\pi \log(p_\pi),$$

where p_π is the relative frequency of π in the time series.

Upshots:

- robust against noise
- invariant under scaling
- computationally efficient and simple.

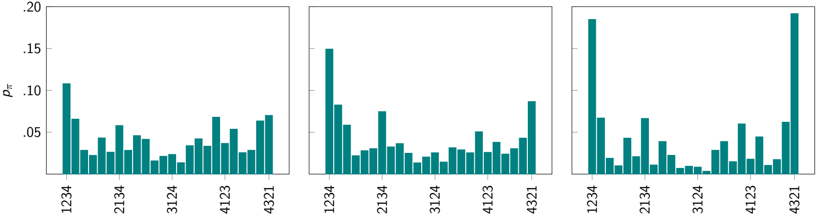


Applications: Detecting pre-seizure activity, sleep stages using EEGs, speech signals, ...

Permutation Entropy as a Shannon Entropy

Re-scale so the permutation entropy of white noise is 1.

$$\mathrm{PE}_n = \frac{-1}{\log(n!)} \sum_{\pi \in \mathcal{S}_n} p_\pi \log(p_\pi)$$



(a) average daily temperature in NYC, (b) daily closing prices of S&P500, and (c) heart rate sampled at .5 second intervals

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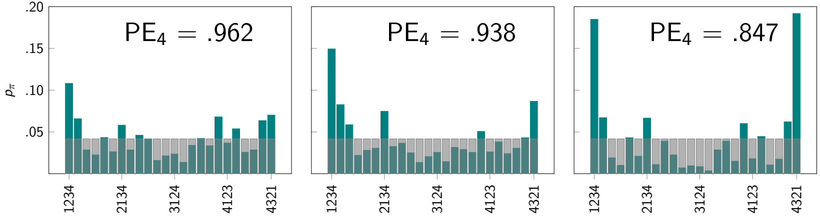
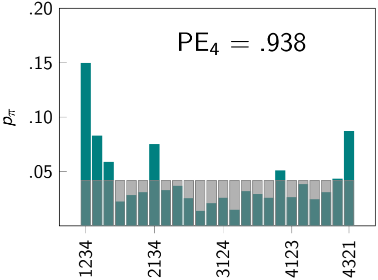


(a) average daily temperature in NYC, (b) daily closing prices of S&P500, and (c) heart rate sampled at .5 second intervals

KL Divergence and Similarity

Given that Z is the true behavior, how surprising is X?

$$\mathrm{D_{KL}}_n(X|Z) = \sum_{\pi \in \mathcal{S}_n} \mathbb{P}_X(\pi) \log \left(\frac{\mathbb{P}_X(\pi)}{\mathbb{P}_Z(\pi)} \right)$$



S&P500 closing prices

For the uniform distribution U,

$$\mathrm{D_{KL}}_n(X|U) = 1 - \mathrm{PE}_n(X).$$

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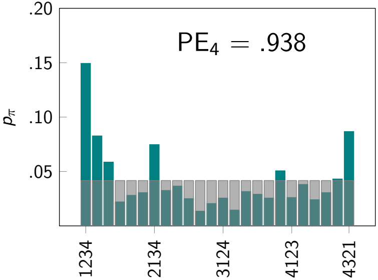

S&P500 closing prices

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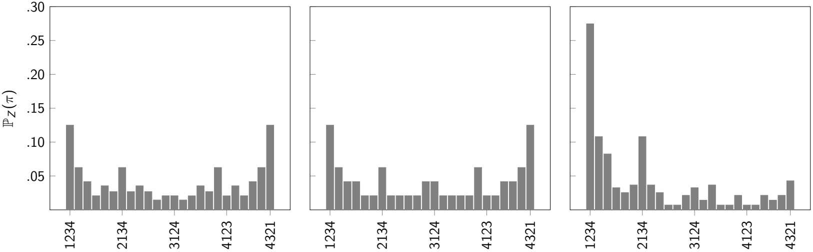
Question: How much of the behavior of the S&P500 is explained by a random walk model?

Random Walks



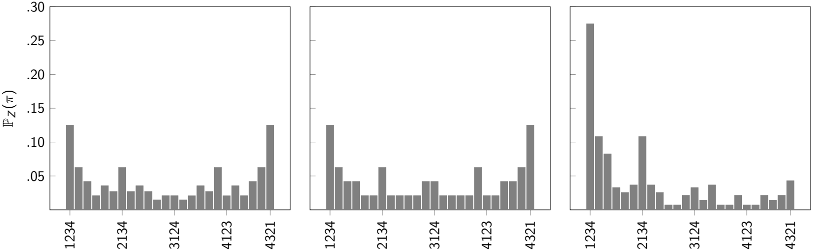
Distribution of patterns in the random walk with steps drawn from (a) standard normal, (b) uniform on $[-.5, .5]$, (c) uniform on $[-.35, .65]$.

Random Walks

For any random walk, Z,

$$\mathbb{P}_Z(\pi) = \mathbb{P}_Z(\pi^{rc}).$$

Theorem (Elizalde and Martinez): Gave a complete combinatorial description of the patterns occurring with the same probability in any random walk.



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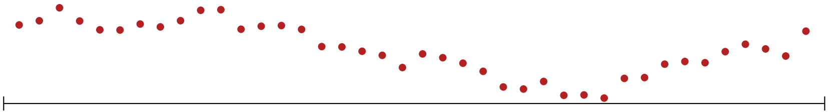
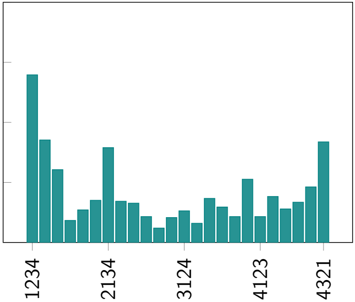
Deviation from a Random Walk Model



Distribution of Patterns in
S&P500 Prices (2012-2017)



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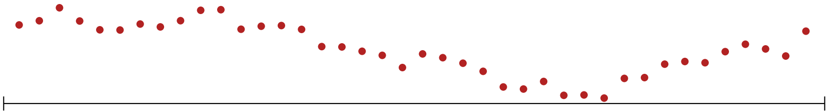

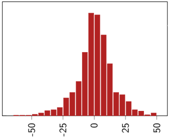

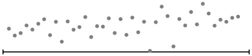

Random Walk
same step distribution

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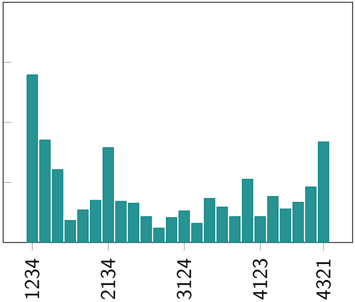

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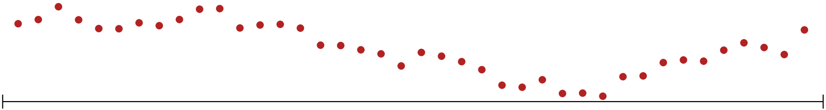

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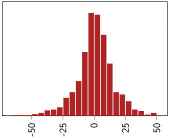

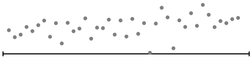

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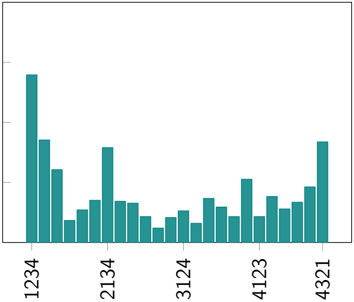

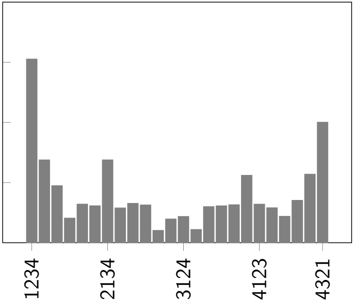

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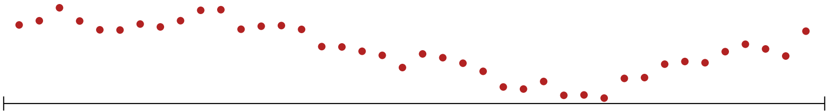

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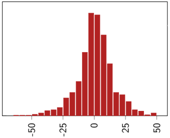

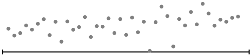

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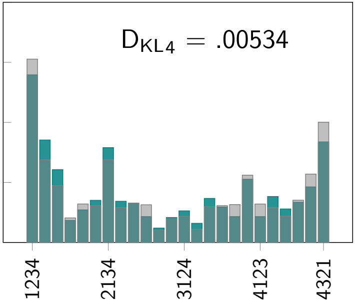

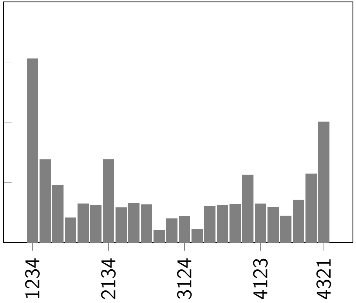

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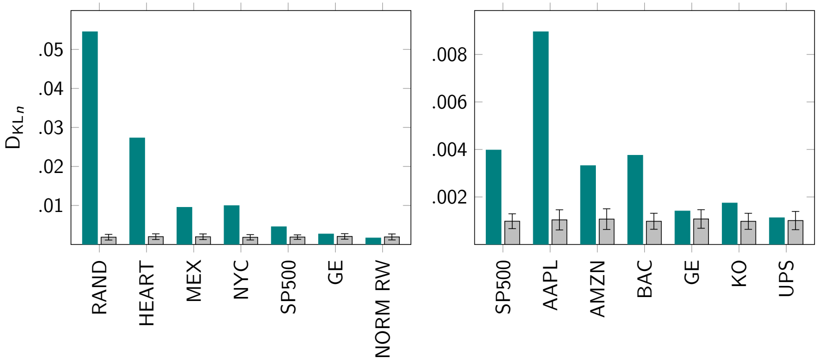
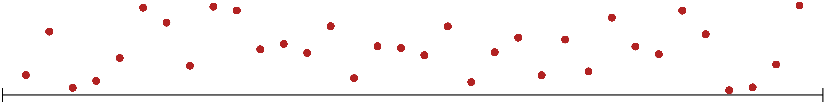
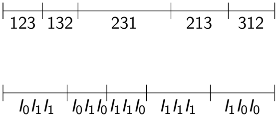
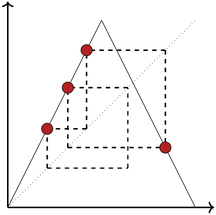


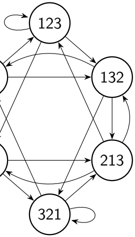
We compute D_{KL_n} for $n = 4$ and the data of length $N = 2000$ (teal).
To test the significance, we generate 400 random walks $\widehat{X}$ of length $N = 2000$ and compute $D_{KL_n}(\widehat{X})$ for each.
The mean and errors are plotted in gray.

Understanding and Extending Pattern-Based Techniques









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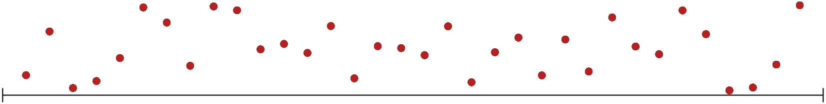
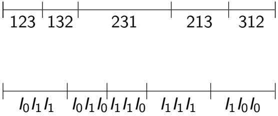
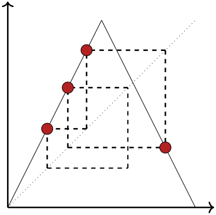
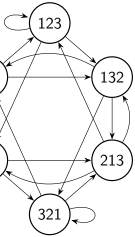


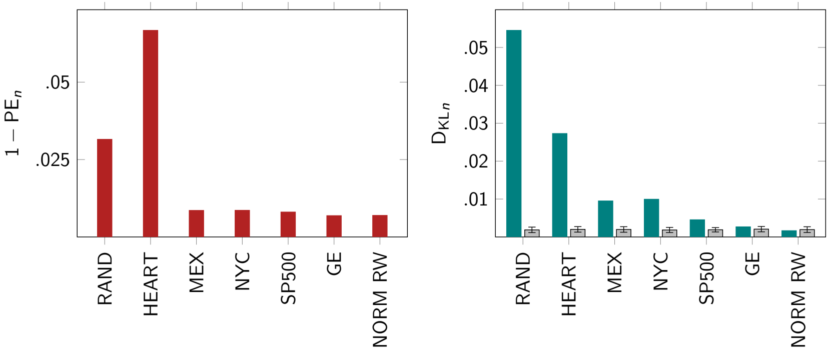






Thank you!

Permutation Entropy of Steps vs Deviation



We measure the independence of steps using permutation entropy (left) for time series of length 2000.

Equivalence Classes of Patterns in Random Walks

Elizalde and Martinez define $\pi \sim \tau$ if $\mathbb{P}_Z(\pi) = \mathbb{P}_Z(\tau)$ for any random walk, Z.

For $n = 3$, the classes are

$$\{123\}, \{132, 213\}, \{213, 132\}, \{321\}.$$

In addition to $\pi \sim \pi^{rc}$, the non-trivial equivalence classes for $n = 4$ and $n = 5$ are

$$\{1432, 2143, 3214\}, \{2341, 3412, 4123\},$$

and

$$\{35412, 52134, 45213, 23541\}, \{23451, 45123, 34512, 51234\},$$

$$\{21534, 23154, 15423, 34215\}.$$

Forbidden Patterns of Up-Down-Up

The smallest forbidden patterns of Up-Down-Up are

$$2134 \quad \text{and} \quad 1324.$$

$$|\ \text{Allow}_4(\text{Up-Down-Up})\ | = 22$$
$$|\ \text{Allow}_5(\text{Up-Down-Up})\ | = 82$$
$$|\ \text{Allow}_6(\text{Up-Down-Up})\ | \approx 300$$