Patterns in Time Series and Dynamical Systems

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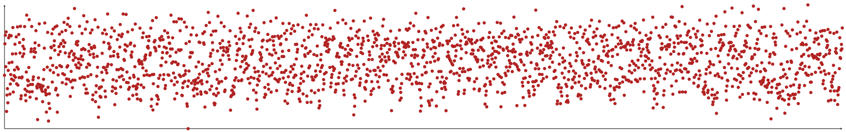
with Daryl DeFord

September 27, 2017

Ten Thousand Foot Overview

We seek a way of measuring unpredictability (i.e. entropy) of time series that is:

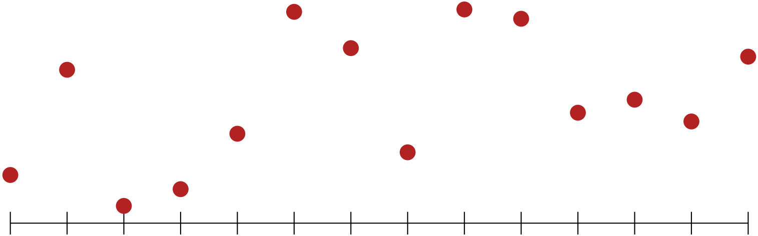
- Robust to noise.
- Without extensive model assumptions.
- Widely applicable.



Examples: EEG and ECG signals (distances between peaks), and financial time series.

Patterns in Time Series

How unpredictable is our time series?



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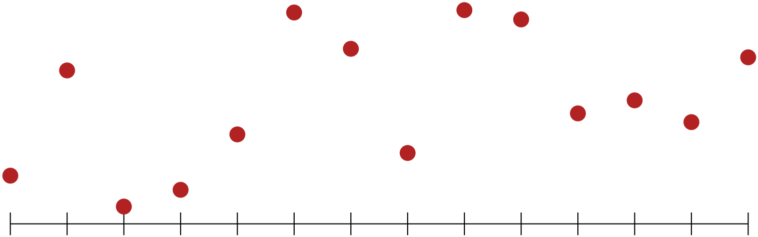

$$\underbrace{(0.21, 0.68, 0.07, 0.15)}_{3412}$$

Patterns: $\{3412\}$

Patterns in Time Series

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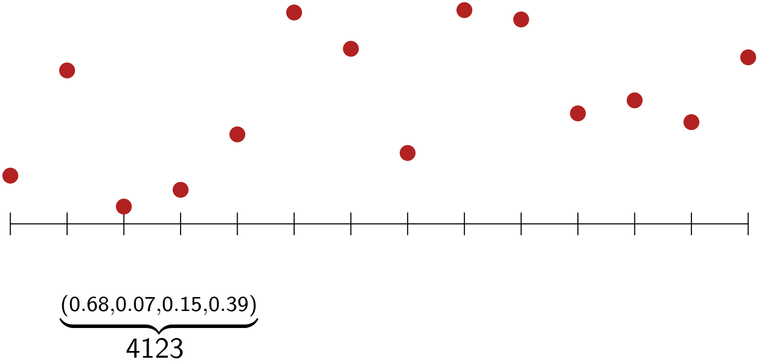

Patterns: $\{3412, 4123\}$

Patterns in Time Series

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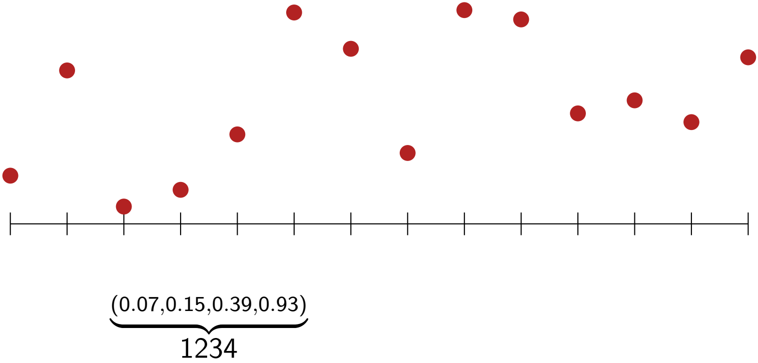
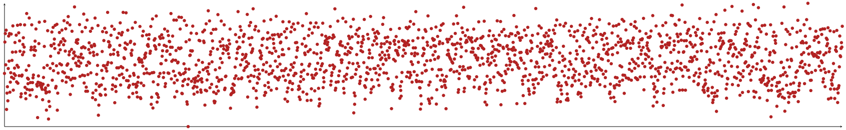


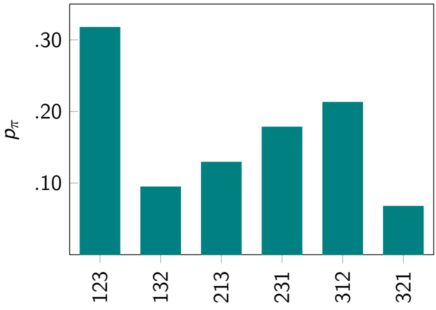
Patterns:

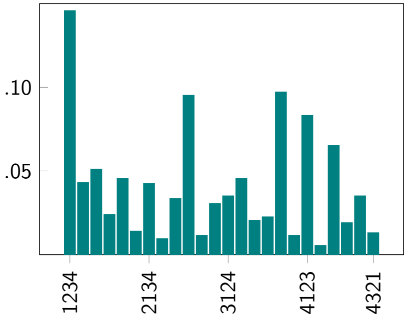
$\{3412, 4123, 1234, 1243, 2431, 4213, 2143, 1432, 4312, 4231, 2314\}$

Patterns in Time Series



The distribution of patterns of length $n = 3$ and $n = 4$.

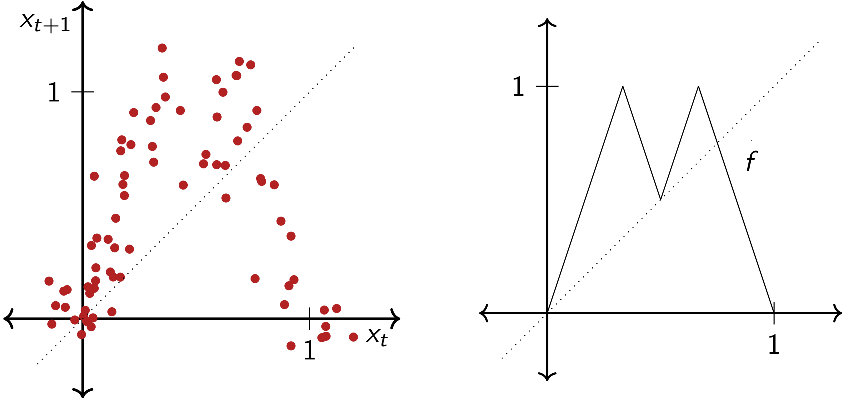




Time Series as Dynamical Systems

We can imagine our time series as being defined by

$$x_{t+1} = f(x_t) + \epsilon.$$

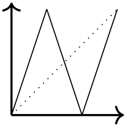


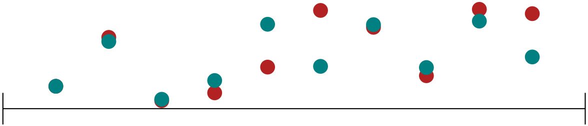
Question: How quickly do errors ruin predictions?

Entropy of a Dynamical System

Goal: Develop meaningful techniques for estimating entropy from time series data.

- Topological entropy. The growth rate of the number of distinguishable time series.





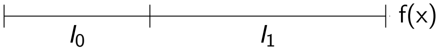
$\{f^t(\mathbf{x_0})\}$ and $\{f^t(\mathbf{x_0})\}$ for $|\mathbf{x_0} - \mathbf{x_0}| = 0.001$

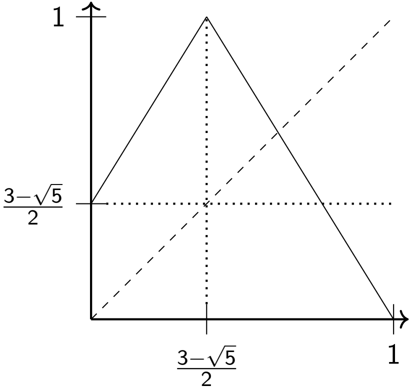
- Kolmogorov-Sinai entropy.

Topological Entropy

Suppose that f is piecewise monotone. Let c_n be the number of monotone segments of f^n, then

$$\text{TopEnt}(f) = \lim_{n\to\infty} \frac{1}{n} \log(c_n).$$





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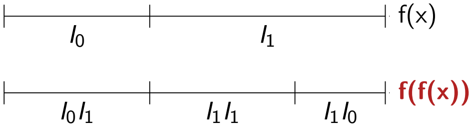

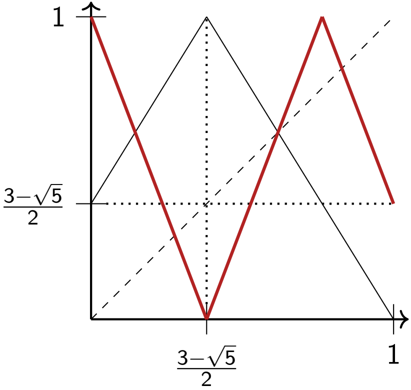

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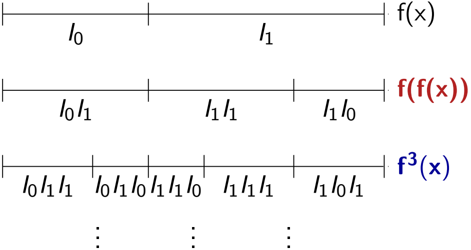

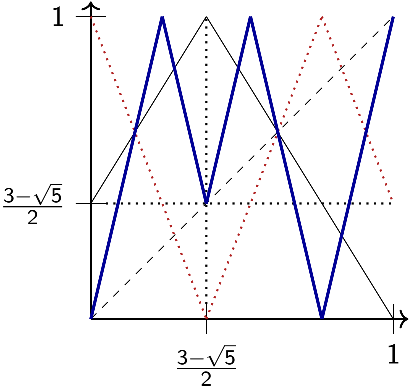

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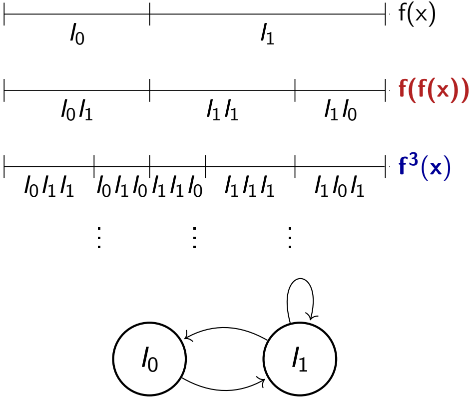

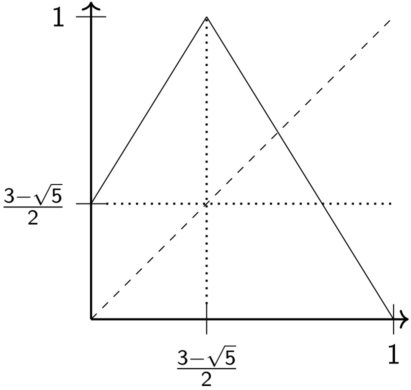

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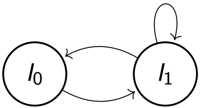
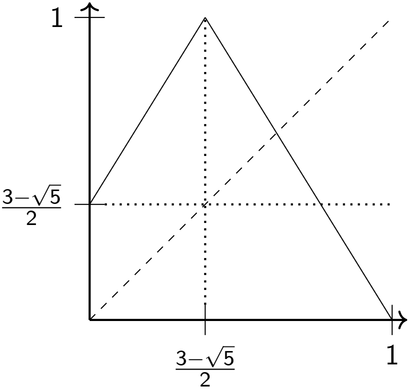


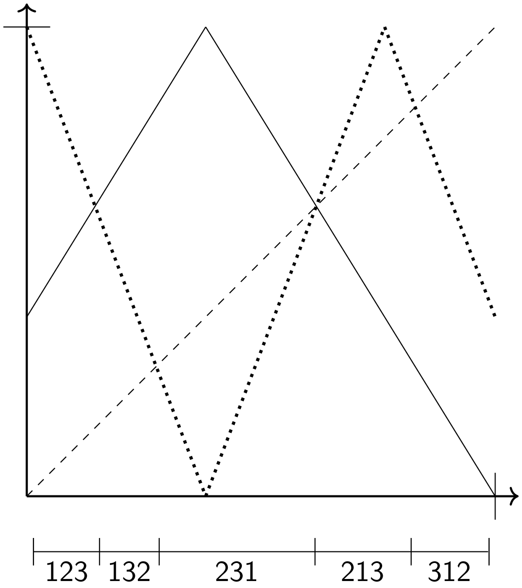
$$\text{Adj}(f) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \to \lambda_1 = \frac{1+\sqrt{5}}{2}$$

and so

$$\boxed{\text{TopEnt}(f) = \log(\tfrac{1+\sqrt{5}}{2})}$$



Patterns in Iterated Functions

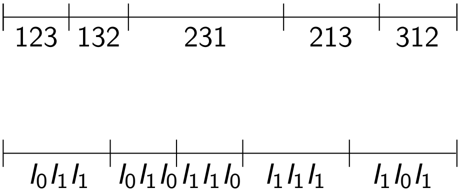


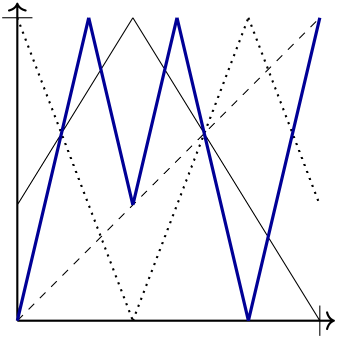
Patterns in Iterated Functions

Theorem: (Bandt, Keller and Pompe) If f is a piecewise monotone map of the interval, then

$$\mathrm{TopEnt}(f) = \lim_{n\to\infty} \frac{1}{n-1} \log(|\,\mathrm{Allow}_n(f)|),$$

where $|\,\mathrm{Allow}_n(f)|$ is the number of distinct patterns of length n.





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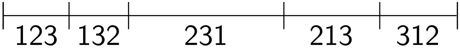
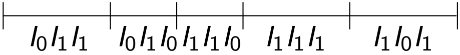
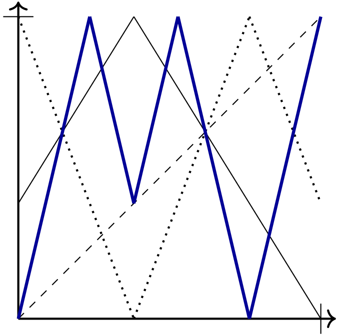




Corollary: f has forbidden patterns.

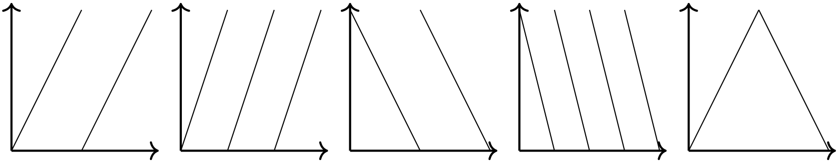


Counting Allowed Patterns

Theorem (Elizalde; Elizalde and M.): We obtained closed formulas for the number of patterns for

$$F_K(x) = Kx \mod 1 \text{ and } G_K(x) = -Kx \mod 1,$$

and reasonable bounds on the allowed patterns of the tent map.



Corollary: The smallest forbidden patterns of

$$F_K(x) = Kx \bmod 1 \text{ and } G_K(x) = -Kx \bmod 1$$

are of length $K + 2$.

Bounding Entropy Using Patterns

Conversely...

Use individual patterns to find best-possible lower bounds on entropy.

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Use individual patterns to find best-possible lower bounds on entropy.

Theorem (Elizalde; Elizalde and M.): Given a pattern π, we determine the infimum topological entropy of a function of the form

$$F_\beta(x) = \beta x \bmod 1 \quad \text{and} \quad G_\beta(x) = -\beta x \bmod 1,$$

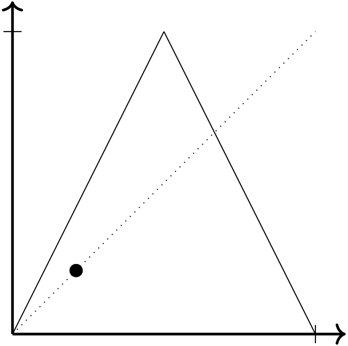
realizing π.

Note that $\mathrm{TopEnt}(F_\beta) = \log(\beta)$ and $\mathrm{TopEnt}(G_\beta) = \log(\beta)$.

The Shape of Patterns

For the tent map, f and $x = .21$.

$$\begin{aligned} & \mathrm{st}(x, f(x), f^2(x), f^3(x), f^4(x)) \\ = \; & \mathrm{st}(.21, -, -, -, -) \end{aligned}$$



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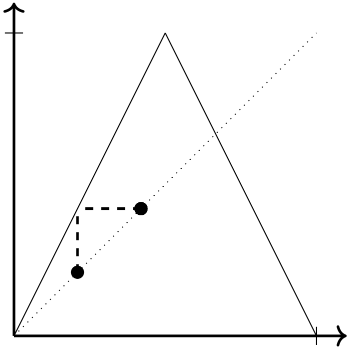

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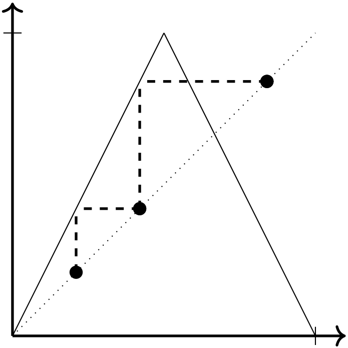

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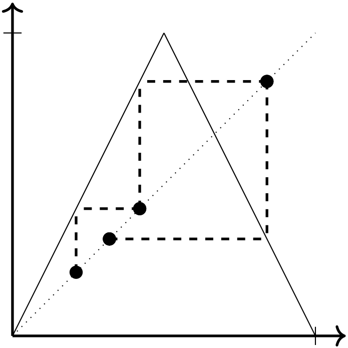

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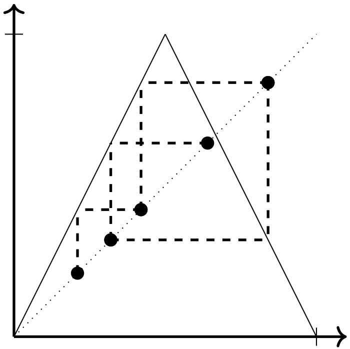

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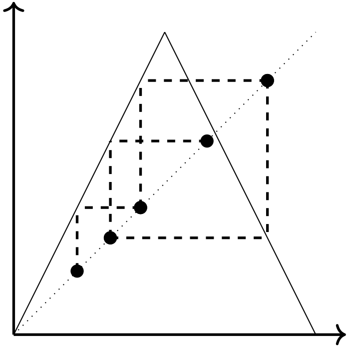
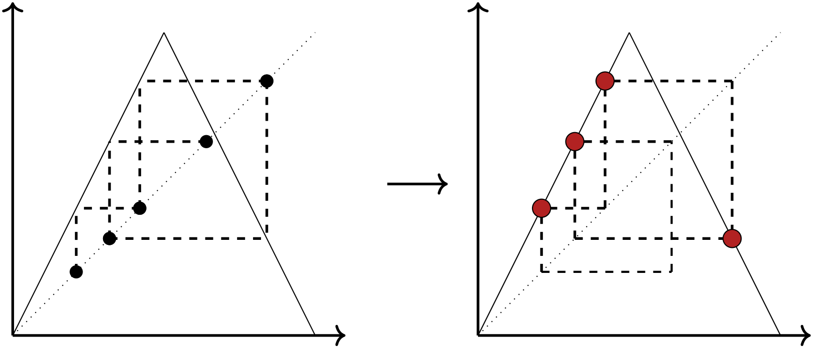
$$\pi = 13524$$



The Shape of Patterns

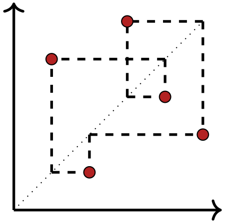
$$\pi = 13524 \longrightarrow \hat{\pi} = (\star, 3, 5, 2, 4) = 345\star2$$



Bounding Topological Entropy

For **continuous** functions, patterns for periodic points give tight lower bounds on topological entropy (Baldwin and others).

Suppose a **continuous** function has a 5-periodic point with pattern $\pi = 52143$ (equiv. $21435, 14352, 43521, 35214$).



Minimal entropy continuous map with a periodic point of shape

$$\hat{\pi} = (5, 2, 1, 4, 3) = 41532$$

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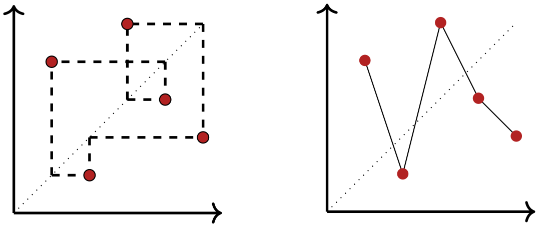

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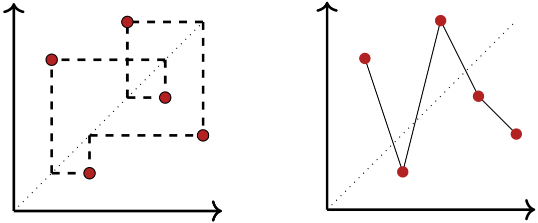

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 = 2.505$$

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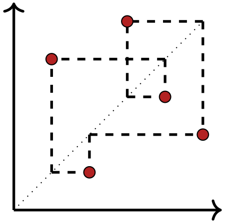

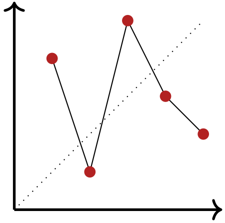

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Minimal entropy continuous map with a periodic point of shape

$\hat{\pi} = (5, 2, 1, 4, 3) = 41532$ is $L_{\hat{\pi}}$ and $\mathrm{TopEnt}(L_{\hat{\pi}}) = \log(2.505)$.

Entropy of a Dynamical System

Goal: Develop meaningful techniques for estimating entropy from time series data.

- Kolmogorov-Sinai entropy: The system will eventually settle, spending a certain proportion of time in each sub-interval, how much unpredictability/surprise remains?

In English:	Random Alphabet:
$\mathbb{P}(Q \to U) \approx .987$,	$\mathbb{P}(A \to B) = 1/26 \approx .038$,
$\mathbb{P}(V \to E) \approx .686$.	$\mathbb{P}(A \to C) = 1/26 \approx .038$.

Kolmogorov-Sinai Entropy

Let μ be an invariant probability measure of a piecewise monotone function $f : I \to I$ (i.e. $\mu(f^{-1}(A)) = \mu(A)$ for measurable sets A).

Definition: Let $\mathcal{P}_n$ be the partition of I into intervals where f^n is monotone. Then

$$h_\mu := \lim_{n\to\infty} \frac{1}{n} \sum_{I_P \in \mathcal{P}_n} -\mu(I_P) \log(\mu(I_P)).$$

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Theorem (Bandt, Keller and Pompe): Let $\mathcal{A}_n$ be the partition of I into sets of the form

$$I_\pi = \{x : \mathrm{st}(x, f(x), \ldots, f^{n-1}(x)) = \pi\}.$$

Then

$$h_\mu = \lim_{n\to\infty} \frac{1}{n-1} \sum_{I_\pi \in \mathcal{A}_n} -\mu(I_\pi)\log(\mu(I_\pi)).$$

Permutation Entropy

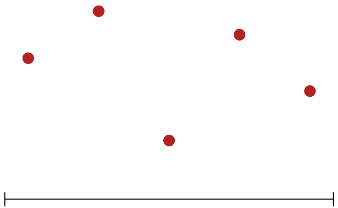
Define the permutation entropy of a time series by

$$\overline{\mathrm{PE}}_n = \frac{-1}{n-1} \sum_{\pi \in \mathcal{S}_n} p_\pi \log(p_\pi),$$

where p_π is the relative frequency of π in the time series.

Upshots:

- robust against noise
- invariant under scaling
- computationally efficient and simple.

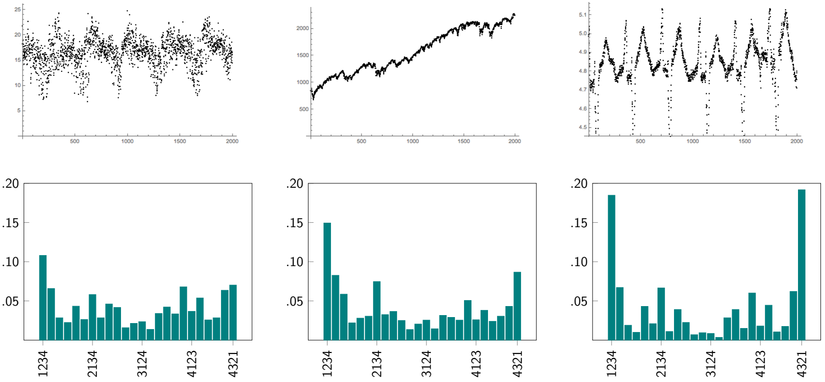


Applications: Detecting pre-seizure activity, sleep stages using EEGs, speech signals, ...

Permutation Entropy as a Shannon Entropy

Re-scale so the permutation entropy of white noise is 1.

$$\text{PE}_n = \frac{-1}{\log(n!)} \sum_{\pi \in \mathcal{S}_n} p_\pi \log(p_\pi)$$



Daily Temp NYC　　Daily S&P500　　Sampled Heart Rate

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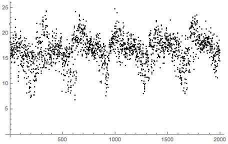
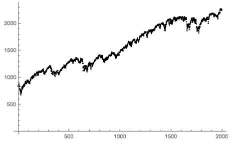
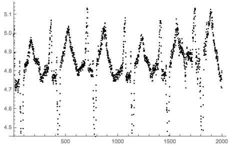
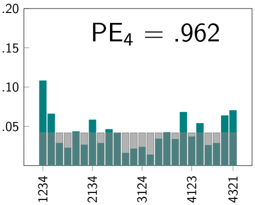
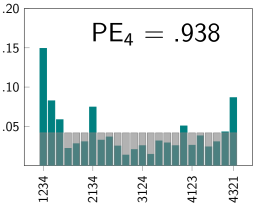




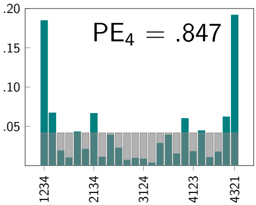




Daily Temp NYC



Daily S&P500

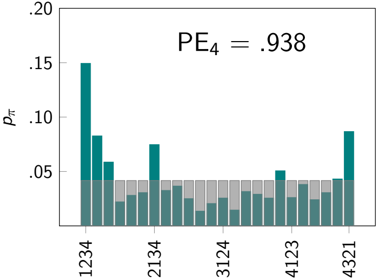


Sampled Heart Rate

KL Divergence and Similarity

Given that Z is the true distribution, how surprising is X?

$$\mathrm{D_{KL}}_n(X|Z) = \sum_{\pi \in \mathcal{S}_n} \mathbb{P}_X(\pi) \log\left(\frac{\mathbb{P}_X(\pi)}{\mathbb{P}_Z(\pi)}\right)$$



S&P500 closing prices

For the uniform distribution U,

$$\mathrm{D_{KL}}_n(X|U) = 1 - \mathrm{PE}_n(X).$$

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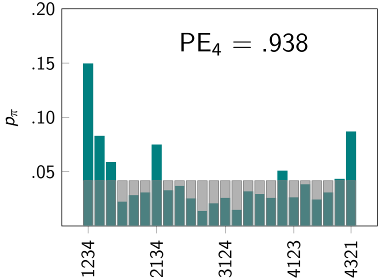

S&P500 closing prices

For the uniform distribution U,

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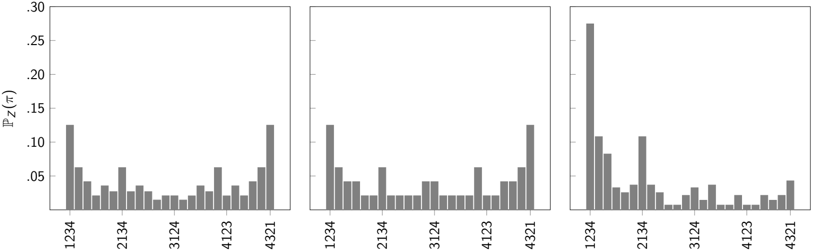
Question: How much of the behavior of the S&P500 is explained by a random walk model?

Patterns in Random Walks



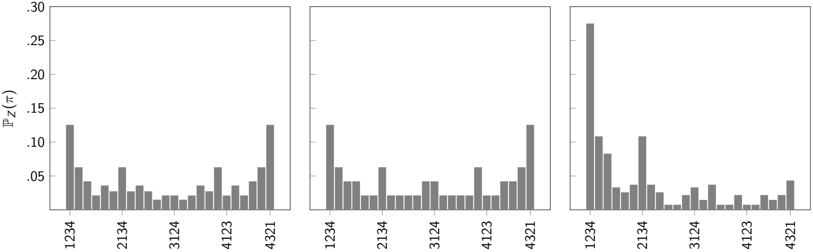
Distribution of patterns in the random walk with steps drawn from (a) standard normal, (b) uniform on $[-.5, .5]$, (c) uniform on $[-.35, .65]$.

Patterns in Random Walks

For any random walk, Z,

$$\mathbb{P}_Z(\pi) = \mathbb{P}_Z(\pi^{rc}).$$

Theorem (Elizalde and Martinez): Gave a complete combinatorial description of the patterns occurring with the same probability in any random walk.



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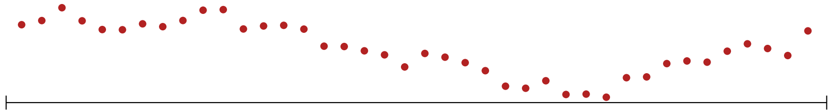
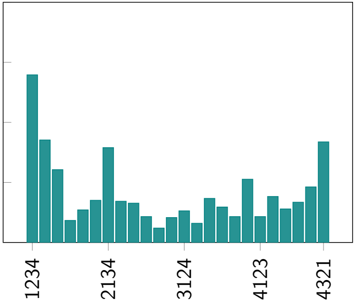
Deviation from a Random Walk Model



Distribution of Patterns in
S&P500 Prices (2012-2017)



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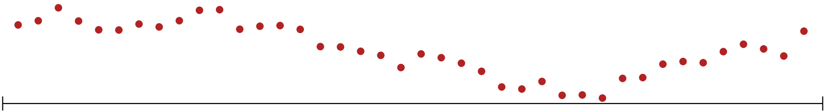

Distribution of Patterns in
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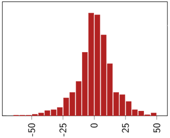
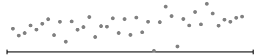
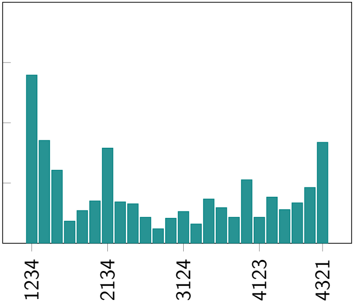
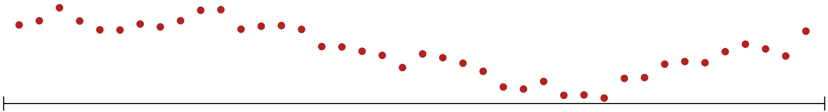




Random Walk
same step distribution



Deviation from a Random Walk Model



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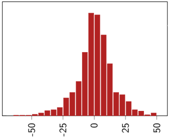

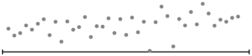

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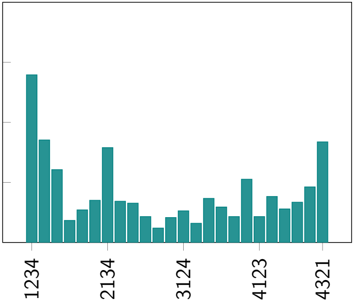

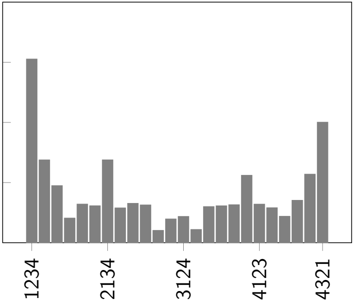

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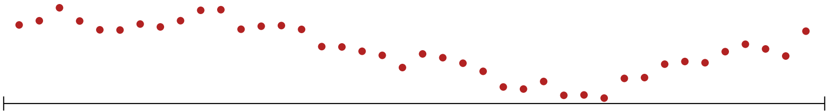

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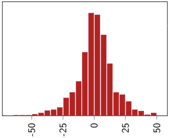

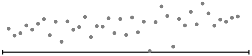

Random Walk
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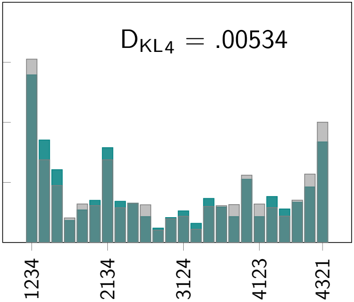
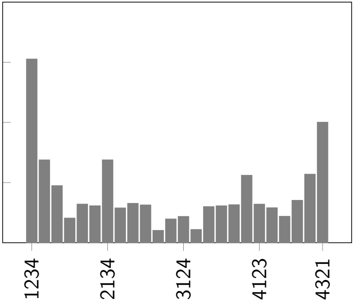
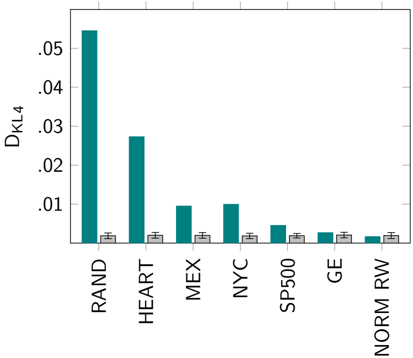


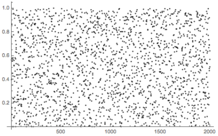


Deviation from a Random Walk

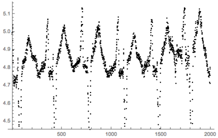
Data of length $N = 2000$.



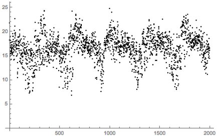
To test the significance, we generate 400 random walks of length $N = 2000$ and compute D_{KL4} for each (in gray).



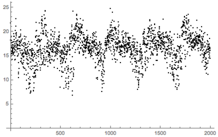
RAND



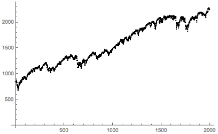
HEART



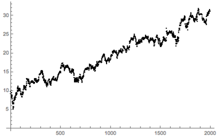
MEX



NYC



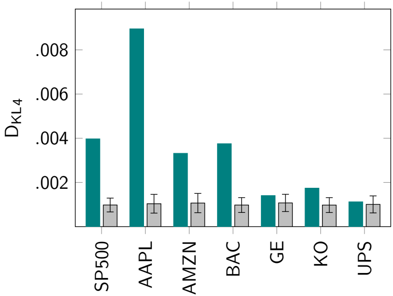
SP500



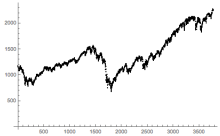
GE

Deviation from a Random Walk: Inefficiency of Markets

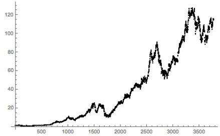
Daily closing prices of stocks 2002-2017 ($N = 3777$)



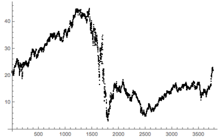
To test the significance, we generate 400 random walks of length $N = 3777$ and compute D_{KL4} for each (in gray).



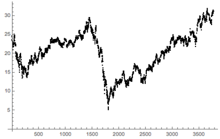
SP500



AAPL



AMZN/BAC



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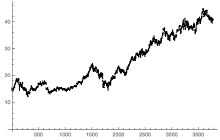
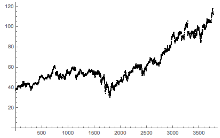
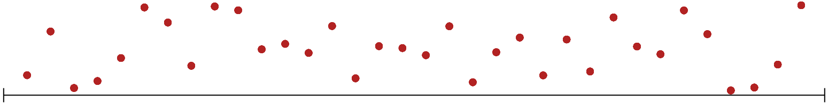


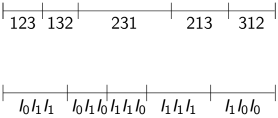
KO

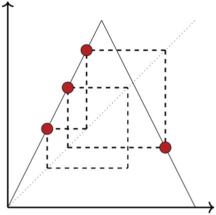


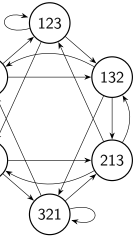
UPS

Understanding and Extending Pattern-Based Techniques

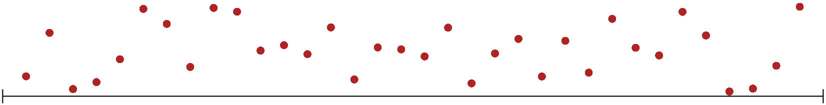


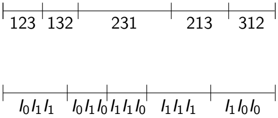


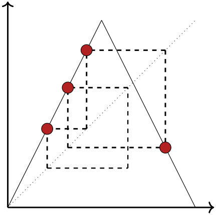


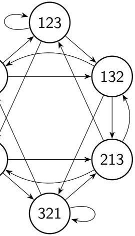


Understanding and Extending Pattern-Based Techniques









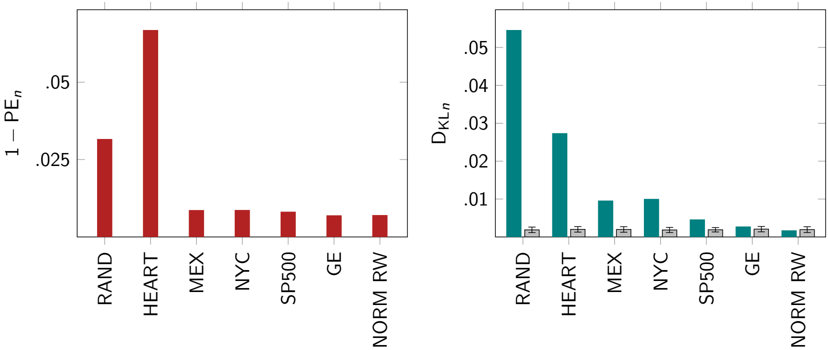
Thank you!

Forbidden Patterns in Time Series

Data	Forbidden Patterns (2000)			Forbidden Patterns (5000)		
	$n=4$	$n=5$	$n=6$	$n=4$	$n=5$	$n=6$
Random	0	0	48	0	0	0
N Rand Walk	0	0	190	0	0	68
Temp NYC	0	0	115	0	0	46
Daily SP500	0	0	199	0	0	58
Sampled Sine	14	106	702	14	106	702
Tent map	12	89	645	12	89	645
$2x$ mod 1	6	72	594	6	72	594
$4x$ mod 1	0	0	181	0	0	90

Notice that Random and the Normal Random Walk contain **all** patterns in the limit.
The true number of forbidden patterns of length $n = 6$ for $F_4(x) = 4x \bmod 1$ is 6.

Permutation Entropy of Steps vs Deviation



We measure the independence of steps using permutation entropy (left) for time series of length 2000.

Equivalence Classes of Patterns in Random Walks

Elizalde and Martinez define $\pi \sim \tau$ if $\mathbb{P}_Z(\pi) = \mathbb{P}_Z(\tau)$ for any random walk, Z.

For $n = 3$, the classes are

$$\{123\}, \{132, 213\}, \{213, 132\}, \{321\}.$$

In addition to $\pi \sim \pi^{rc}$, the non-trivial equivalence classes for $n = 4$ and $n = 5$ are

$$\{1432, 2143, 3214\}, \{2341, 3412, 4123\},$$

and

$$\{35412, 52134, 45213, 23541\}, \{23451, 45123, 34512, 51234\},$$

$$\{21534, 23154, 15423, 34215\}.$$

Minimal Forbidden

The smallest forbidden patterns of $F_4(x) = 4x \bmod 1$ are

$$123456, 654321, 123465, 654312.$$

The smallest forbidden patterns of $G_4(x) = -4x \bmod 1$ are

$$615243, 324156, 342516, 162534, 453621, 435261.$$