Patterns in Dynamical Systems

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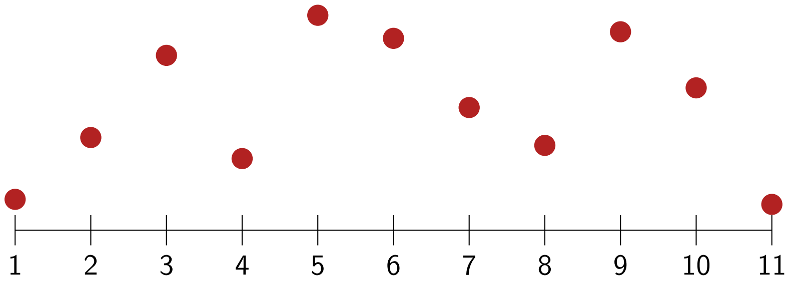
Patterns in Time Series

What is the distribution of patterns appearing in our time series?



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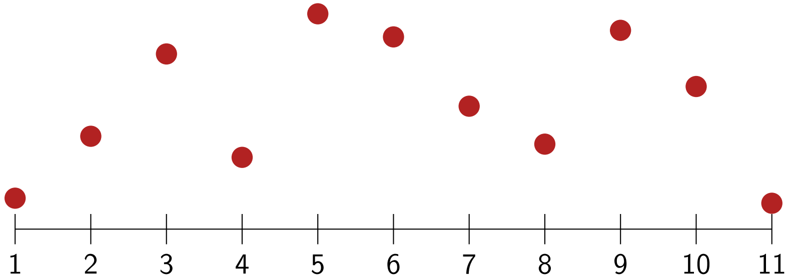

$$\underbrace{(0.14, 0.41, 0.77, 0.32)}_{1342}$$

Patterns: $\{1342\}$

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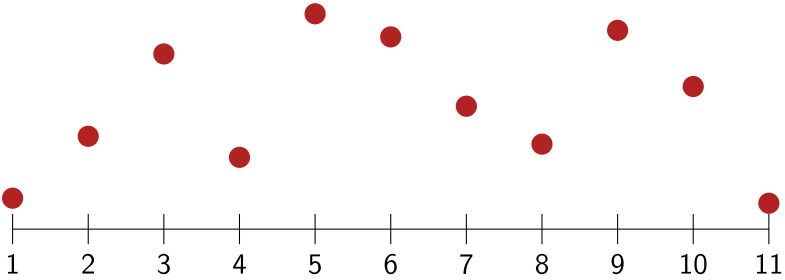

$$\underbrace{(0.41, 0.77, 0.32, 0.95)}_{2314}$$

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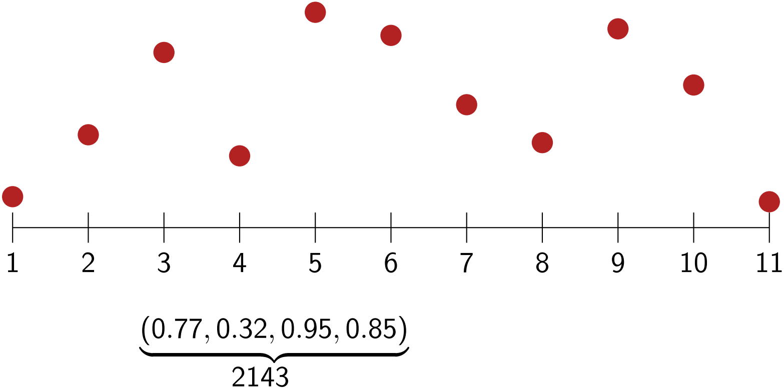

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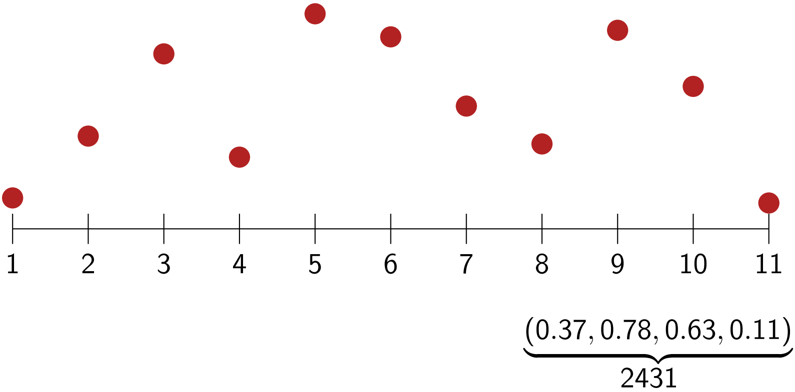


Patterns: $\{1342, 2314, 2143, 1432, 4321, 3214, 2143, 2431\}$

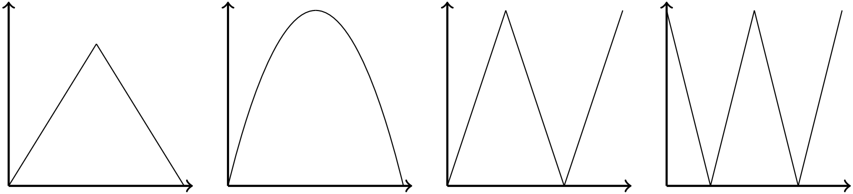
Dynamical Complexity

Informally, we can measure the complexity (i.e. unpredictability) of a time series by considering the function f such that

$$x_t \approx a \longrightarrow x_{t+1} \approx f(a)$$

That is, we view our time-series as being defined by iteration:

$$x_{t+1} = f(x_t) + \epsilon$$



Topological entropy:

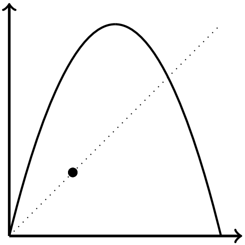
a) $\log(\frac{1+\sqrt{5}}{2})$ b) $\log(2)$ c) $\log(3)$ d) $\log(4)$

Function Iteration

Example:

Let $f(x) = 4x(1-x)$. Then

$$(x, f(x), f^2(x), f^3(x)) = (.30, -, -, -)$$



Function Iteration

Example:

Let $f(x) = 4x(1-x)$. Then

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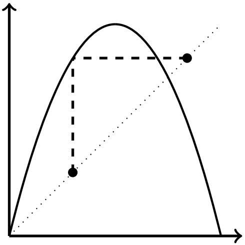

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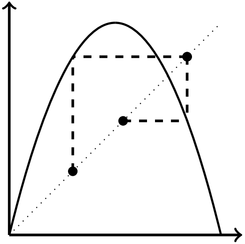
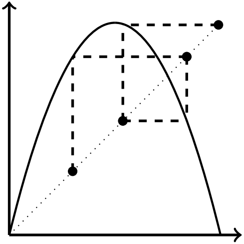


Function Iteration

Example:

Let $f(x) = 4x(1 - x)$. Then

$(x, f(x), f^2(x), f^3(x)) = (.30, .84, .53, .99)$



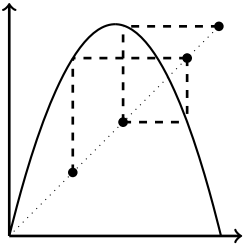
Function Iteration

Example:

Let $f(x) = 4x(1-x)$. Then

$(x, f(x), f^2(x), f^3(x)) = (.30, .84, .53, .99)$
and so

$$\text{Pat}(.3, f, 4) = \text{st}(.30, .84, .53, .99) = 1324$$



How is the complexity (i.e. topological entropy) of f reflected in patterns?

Patterns and Dynamical Systems

Theorem (Bandt-Keller-Pompe): Every piecewise-monotone map $f : [0,1] \to [0,1]$ has forbidden patterns, i.e. patterns that never arise as iterates.

$$\# \text{AllowedPatterns}_n(f) \sim k^n \longleftrightarrow \text{TopEntropy}(f) = \log(k)$$

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Example: Let $f(x) = 4x(1-x)$.

$$321 \notin \mathsf{Allow}(f) \to \underbrace{4321, 1432, 54213,}_{\text{contain consecutive 321}} \ldots \notin \mathsf{Allow}(f)$$

In practice, short time series are problematic,

but we can take a different perspective.

Sarkovskii's Theorem

An *n−periodic point* of a map is a point such that

$$f^n(x) = x \text{ and } f^i(x) \neq x \text{ for all } 1 \leq i < n.$$

Theorem (Sarkovskii):
If a continuous map f of the unit interval has an m-periodic point and $\ell \lhd m$ in the Sarkovskii ordering

$$1 \lhd 2 \lhd 2^2 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 5{\cdot}2^n \lhd 3{\cdot}2^n \lhd \cdots \lhd 7{\cdot}2 \lhd 5{\cdot}2 \lhd 3{\cdot}2 \lhd \cdots \lhd 7 \lhd 5 \lhd 3$$

then f must also have an ℓ-periodic point.

Question: Is there a similar order for the permutation structure of periodic points?

Cycle Type

Let x be a periodic point of order n and $\mathrm{Pat}(x, f, n) = \pi$.

The *cycle type* of x is $\hat{\pi} \in \mathcal{C}_n$ where

$$\pi = \pi_1 \pi_2 \dots \pi_n \to \hat{\pi} = (\pi_1, \pi_2, \dots, \pi_n).$$

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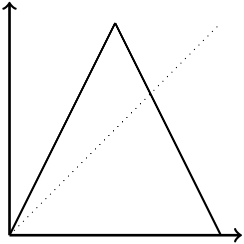
Example: Consider the tent map, Λ.



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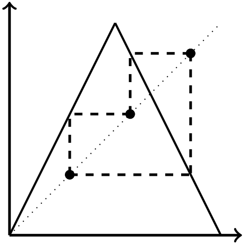
Example: Consider the tent map, Λ.

A 3-periodic orbit of Λ is:

$$(x, \Lambda(x), \Lambda^2(x)) = \left(\frac{6}{7}, \frac{2}{7}, \frac{4}{7}\right)$$

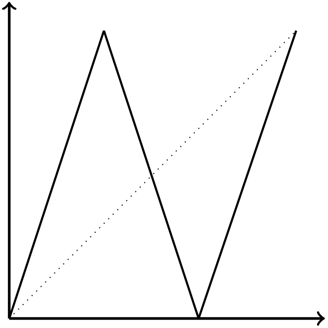


Giving $\text{Pat}(\frac{6}{7}, \Lambda, 3) = 312$ and

$$\hat{\pi} = (3, 1, 2) = 231$$

The Shape of Cycles

The representative of a 6-periodic orbit of H is $x = \frac{21}{26}$.

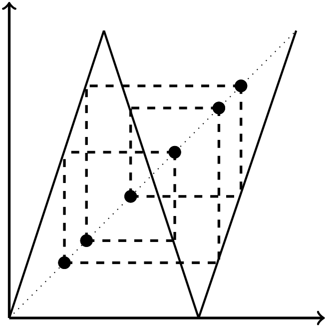


The Shape of Cycles

The representative of a 6-periodic orbit of H is $x = \frac{21}{26}$.

$$\mathrm{Pat}(x, H, 6) = \mathrm{st}\left(\frac{21}{26}, \frac{11}{26}, \frac{19}{26}, \frac{5}{26}, \frac{15}{26}, \frac{7}{26}\right) = 635142$$

The *cycle type* of the orbit is $\hat{\pi} = (6, 3, 5, 1, 4, 2) = 465213$.



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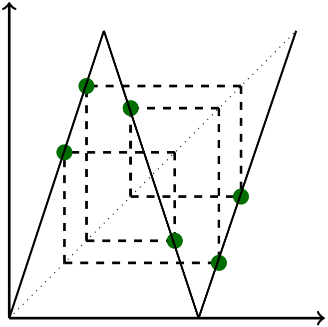

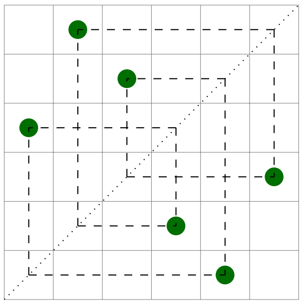

$\hat{\pi} = 465213$

Forcing Order

A cycle $\hat{\pi}$ *forces* a cycle $\hat{\tau}$ if, for any continuous map f, whenever $\hat{\pi} \in \text{AlCyc}(f)$ then $\hat{\tau} \in \text{AlCyc}(f)$ as well. [Baldwin]

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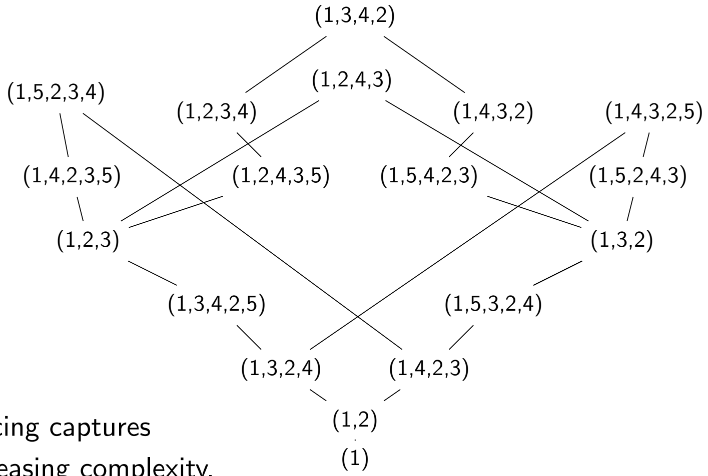
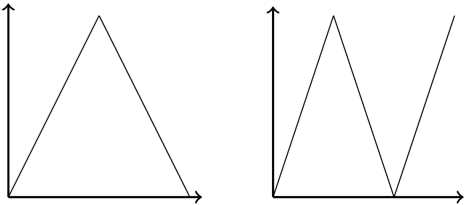
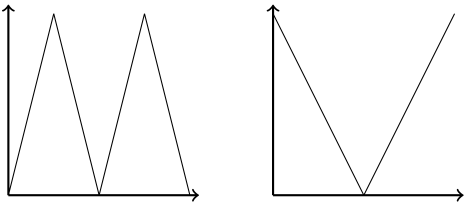


Forcing captures increasing complexity.

Alternating Signed Shifts



The graphs of M_σ for $\sigma = +-$, $\sigma = + - +$.



The graphs of M_σ for $\sigma = + - +-$, and $\sigma = -+$, respectively.

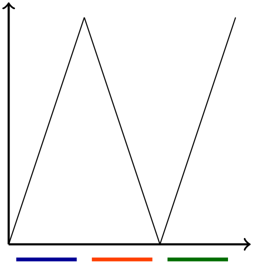
Itineraries

Name the monotonic intervals of M_σ: $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$

$$(x, M_\sigma(x), M_\sigma^2(x), \ldots) \;=\; (.136, .409, .772, .316, .949, .847, \ldots)$$
$$\rightarrow \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{0} \quad \mathbf{2} \quad \mathbf{2} \ldots$$

And $v_1 v_2 v_3 \ldots <_\sigma w_1 w_2 w_3 \ldots$ if

- $v_1 < w_1$
- $\mathbf{v_1 = w_1 = 0}$ and $v_2 v_3 \ldots <_\sigma w_2 w_3 \ldots$
- $\mathbf{v_1 = w_1 = 1}$ and $v_2 v_3 \ldots >_\sigma w_2 w_3 \ldots$
- $\mathbf{v_1 = w_1 = 2}$ and $v_2 v_3 \ldots <_\sigma w_2 w_3 \ldots$



And so

$$\mathrm{Pat}(01202..., M_\sigma, 4) = \mathrm{st}(01202..., 1202..., 202..., 02...) = 1342$$

Segmentations

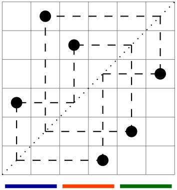
Ask: Is $\hat{\pi}$ a cycle of M_σ?

An σ-segmentation of $\hat{\pi}$ is a sequence $0 = e_0 \leq e_1 \leq \cdots \leq e_N = n$ where each segment $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2}\ldots\hat{\pi}_{e_{t+1}}$ has the monotonicity σ_t.

A $+-+$-segmetation of

$$\hat{\pi} = (6,4,1,3,5,2) = 365124$$

3 6 | 5 1 | 2 4



From this, define a word ω by

$$\pi = \; 6 \quad 4 \quad 1 \quad 3 \quad 5 \quad 2$$
$$\omega = \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_$$

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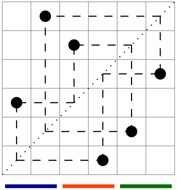

From this, define a word ω by

$$\begin{array}{lcccccc} \pi = & 6 & 4 & 1 & 3 & 5 & 2 \\ \omega = & \_\_ & \_\_ & \mathbf{0} & \_\_ & \_\_ & \mathbf{0} \end{array}$$

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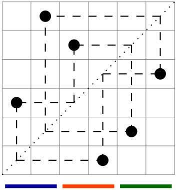

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$$\pi = \ 6 \quad 4 \quad 1 \quad 3 \quad 5 \quad 2$$
$$\omega = \ \_\_ \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \_\_ \quad \mathbf{0}$$

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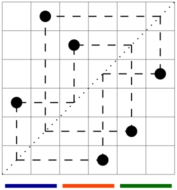


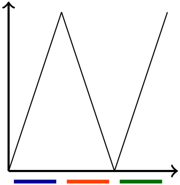
From this, define a word ω by

$$\pi = 6 \quad 4 \quad 1 \quad 3 \quad 5 \quad 2$$
$$\omega = \mathbf{2} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{0}$$

Segmentations

And $v_1 v_2 v_3 \ldots <_\sigma w_1 w_2 w_3 \ldots$ if $v_1 < w_1$ or

- $v_1 = w_1 \in \{\mathbf{0}, \mathbf{2}\}$ and $v_2 v_3 \ldots <_\sigma w_2 w_3 \ldots$
- $v_1 = w_1 = \mathbf{1}$ and $v_2 v_3 \ldots >_\sigma w_2 w_3 \ldots$



Using the order inherited from M_σ, the word

$$\omega^\infty = (210120)^\infty$$

is a 6-periodic point inducing $\pi = 641352$ and cycle type

$$\hat{\pi} = (6, 4, 1, 3, 5, 2)$$

Theorem (Baldwin; Archer and Elizalde): $\hat{\pi} \in \text{AlCyc}(M_\sigma)$ if and only if $\hat{\pi}$ has a σ-segmentation.

Patterns at Any Point

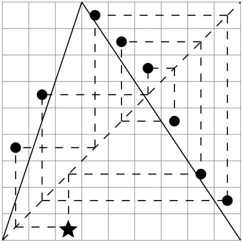
Define a bijection $\mathcal{S}_n$ to marked cycles $\mathcal{C}_n^\star$ by

$$\pi = \pi_1\pi_2\ldots\pi_n \to \hat{\pi}^\star = (\star, \pi_2, \ldots, \pi_n)$$

Theorem (Archer, Elizalde, M.): $\pi \in \mathsf{Allow}(M_\sigma)$ if and only if $\hat{\pi}^\star$ has a σ-segmentation.
(That satisfies a certain technical condition it nearly always does.)

Example:

$$\pi = 149267583 \mapsto \hat{\pi} = (\star, 4, 9, 2, 6, 7, 5, 8, 3) = 46\star987532$$



References

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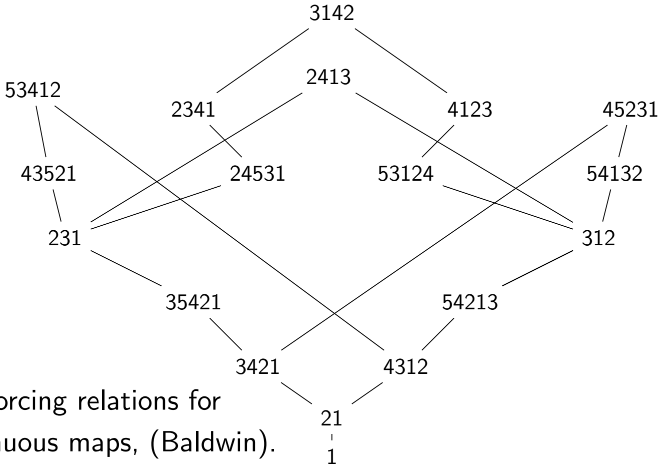
[2] K. Archer, S. Elizalde and K. Moore, "Patterns of Signed Shifts and Negative Shifts", *Discrete Math. Theor. Comput. Sci. proc.,* to appear. (Extended Abstract)

[3] C. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations, *Nonlinearity* 15 (2002), 1595–1602.

[4] S. Baldwin, Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions *Discrete Mathematics* 76 (1987), 111–127.

Forcing Order

For a family of interval maps $\mathcal{F}$, a cycle $\hat{\pi}$ *forces* a cycle $\hat{\tau}$ if, for any $f \in \mathcal{F}$, if $\hat{\pi} \in \mathrm{AlCyc}(f)$ then $\hat{\tau} \in \mathrm{AlCyc}(f)$ as well.



The forcing relations for continuous maps, (Baldwin).
One line notation for cycles.