Patterns of Negative Shifts and Signed Shifts

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Patterns in Dynamical Systems

Let $f : X \mapsto X$ be a map of a linearly ordered set X. For any point $x \in X$, consider the finite sequence

$$x, f(x), f(f(x)), f^3(x), \ldots, f^{n-1}(x),$$

and associate a permutation to this sequence determined by the relative order, an *allowed pattern*. We write

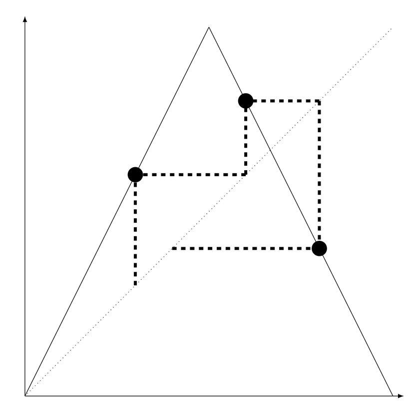
$$\mathrm{Pat}(x, f, n) = \mathrm{st}(x, f(x), f(f(x)), \ldots, f^{n-1}(x)) = \pi.$$

Example: Consider the tent map

$$f(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

$$\mathrm{Pat}(.3, f, 4) = \mathrm{st}(.3, .6, .8, .4) = 1342$$

Moreover, one can show $321 \notin \mathrm{Allow}(f)$.



Cobweb plot for $x = .3$.

Forbidden Patterns

Theorem ([2]): Every piecewise-monotone map $f : [0,1] \mapsto [0,1]$ has forbidden patterns, i.e. patterns that never arise as iterates. Moreover, the number of allowed patterns is asymptotic to k^n, where the topological entropy of f is equal to $\log(k)$.

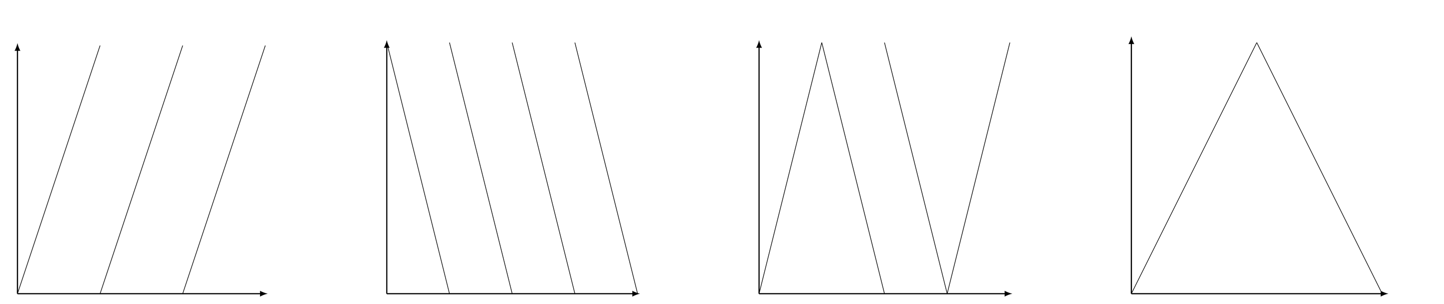
Signed Shifts

The signature of a signed shift is $\sigma = \sigma_0\sigma_1 \ldots \sigma_{k-1} \in \{+,-\}^k$. Let $\mathcal{W}_k = \{0, 1, \ldots, k-1\}^\infty$ be given with the linear order $<_\sigma$ by defining $v_1v_2v_3 \ldots <_\sigma w_1w_2w_3 \ldots$ if

- $v_1 < w_1$,
- $v_1 = w_1$, $\sigma_{v_1} = +$ and $v_2v_3 \ldots <_\sigma w_2w_3 \ldots$, or
- $v_1 = w_1$, $\sigma_{v_1} = -$ and $v_2v_3 \ldots >_\sigma w_2w_3 \ldots$.

We define the *signed shift* $\Sigma_\sigma : (\mathcal{W}_k, <_\sigma) \mapsto (\mathcal{W}_k, <_\sigma)$ as the map

$$\Sigma(w_1w_2w_3 \ldots) = w_2w_3 \ldots.$$



$\sigma = +++$ $\quad$ $\sigma = ----$ $\quad$ $\sigma = +--+$ $\quad$ $\sigma = +-$

Denote by Σ_k and Σ_{-k} shifts with $\sigma = +^k$ and $\sigma = -^k$, respectively.

Example: $\mathrm{Pat}(2010^\infty, \Sigma_3, 4) = \mathrm{st}(2010^\infty, 010^\infty, 10^\infty, 0^\infty) = 4231$
$\mathrm{Pat}(11010^\infty, \Sigma_{+-}, 5) = \mathrm{st}(11010^\infty, 1010^\infty, 010^\infty, 10^\infty, 0^\infty) = 34251$

A Bijection and Segmentations

Define a bijection from $\mathcal{S}_n$ to marked cycles of length n by

$$\pi = \pi_1\pi_2 \ldots \pi_n \mapsto \hat{\pi} = (\star, \pi_2, \pi_3, \ldots, \pi_n) = \hat{\pi}_1\hat{\pi}_2 \ldots \hat{\pi}_n.$$

A *σ-segmentation* of $\hat{\pi}$ is $0 = e_0 \le \cdots \le e_k = n$ such that $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2} \ldots \hat{\pi}_{e_{t+1}}$ is increasing if $\sigma_t = +$ and decreasing if $\sigma_t = -$ (some endpoint conditions omitted).

A σ-segmentation of $\hat{\pi}$ defines a finite word $\zeta = z_1z_2 \ldots z_{n-1}$, by taking $z_i = j$ whenever $e_j < \pi_i \le e_{j+1}$, for $1 \le i \le n-1$.

When $\pi_n \ne n$, let $\pi_x = \pi_n + 1$ and $p = z_{[x,n-1]}$.
When $\pi_n \ne 1$, let $\pi_y = \pi_n - 1$ and $q = z_{[y,n-1]}$.

Characterization of Allowed Patterns

Theorem: $\pi \in \mathrm{Allow}(\Sigma_\sigma)$ if and only if there is a σ-segmentation of $\hat{\pi}$ such that $p \ne q^2$ and $q \ne p^2$.

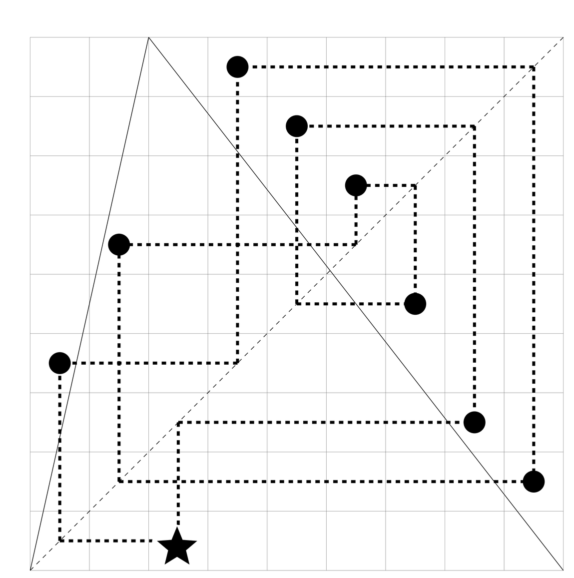
Informally, $\pi \in \mathrm{Allow}(\Sigma_\sigma)$ if and only if $\hat{\pi}$ has the same shape as Σ_σ.

Example: $\pi = 149267583 \mapsto \hat{\pi} = (\star, 4, 9, 2, 6, 7, 5, 8, 3) = 46\star987532$
We take a $(+-)$-segmentation of $\hat{\pi}$ to be $(e_0, e_1, e_2) = (0, 2, 9)$ giving

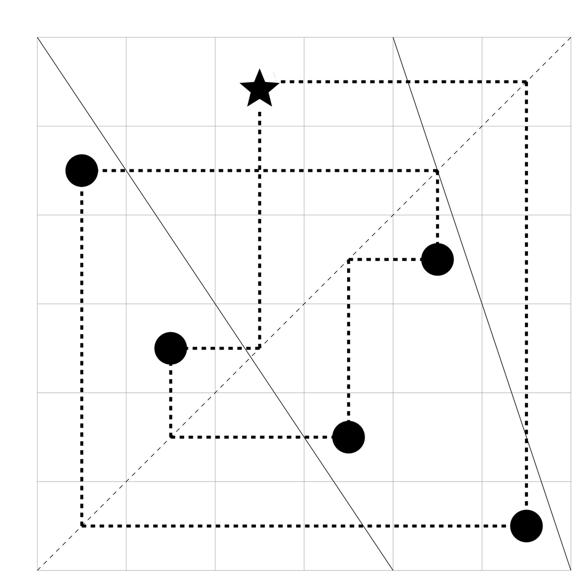
$$\zeta = 01101111, q = z_{[4,8]} = 01111 \text{ and } p = z_{[2,8]} = 1101111.$$

Therefore, $\pi \in \mathrm{Allow}(\Sigma_{+-})$; and whenever $m \ge 4$, for $s^{(m)} = \zeta p^{2m}10^\infty$

$$\mathrm{Pat}(s^{(m)}, \Sigma_{+-}, 9) = \pi.$$



Cycle diagram: $\hat{\pi} = 46\star987532$.



Cycle diagram: $\hat{\pi} = 53\star241$.

Example: $\pi = 615423 \mapsto \hat{\pi} = (\star, 1, 5, 4, 2, 3) = 53\star241$ We take a $(--)$-segmentation of $\hat{\pi}$ to be $(e_0, e_1, e_2) = (0, 4, 6)$ giving

$$\zeta = 10100, q = 0, p = 00, \text{ and so } p = q^2.$$

Although $\hat{\pi}$ has a $(--)$-segmentation, $\pi \notin \mathrm{Allow}(\Sigma_{-2})$.
A -3-segmentation of $\hat{\pi}$ is $(e'_0, e'_1, e'_2, e'_3) = (0, 2, 4, 6)$, defining the prefix

$$\zeta' = 20210, q' = 0 \text{ and } p' = 10.$$

Since $p' \ne (q')^2$, we have $\pi \in \mathrm{Allow}(\Sigma_{-3})$.

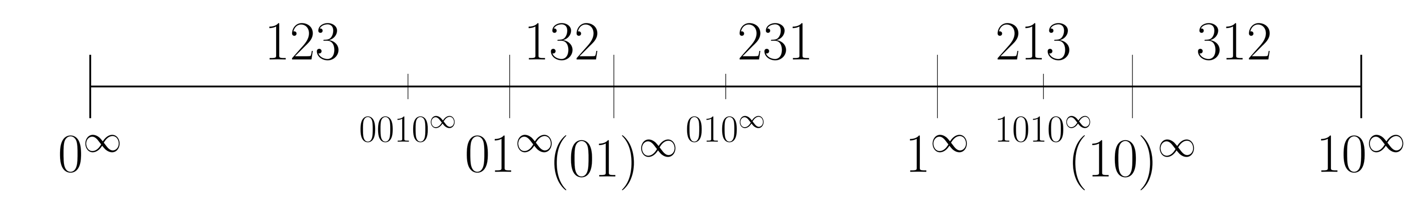
Allowed Intervals

Theorem: Let $\mathrm{Int}(\pi, \Sigma_\sigma) = \{w : \mathrm{Pat}(w, \Sigma_\sigma, n) = \pi\}$. Then $\mathrm{Int}(\pi, \Sigma_\sigma)$ is a finite union of the intervals of the form

$$\{\zeta w_{[n,\infty)} : q^\infty <_\sigma w_{[n,\infty)} <_\sigma p^\infty\}$$

where ζ, q and p are defined (when possible) by a valid σ-segmentation of $\hat{\pi}$.

Example: The intervals $\mathrm{Int}(\pi, \Sigma_{+-})$ for $\pi \in \mathrm{Allow}_3(\Sigma_{+-})$.



Enumerations

$N(\pi) := \min\{k : \pi \in \mathrm{Allow}(\Sigma_k)\}$ and $\overline{N}(\pi) := \min\{k : \pi \in \mathrm{Allow}(\Sigma_{-k})\}$

Theorem ([4]): $N(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \delta(\hat{\pi})$,
where $\delta(\hat{\pi})$ is usually 0 and sometimes 1.
Theorem ([3], [5]): $\overline{N}(\pi) = 1 + \mathrm{asc}(\hat{\pi}) + \epsilon(\hat{\pi})$,
where $\epsilon(\hat{\pi})$ is usually 0 and sometimes 1.

$n \setminus k$	2	3	4	5	6
3	6				
4	18	6			
5	48	66	6		
6	126	402	186	6	
7	306	2028	2232	468	6

$|\{\pi \in \mathcal{S}_n : N(\pi) = k\}|$

$n \setminus k$	2	3	4	5	6
3	6				
4	20	4			
5	54	62	4		
6	140	408	168	4	
7	336	2084	2196	412	4

$|\{\pi \in \mathcal{S}_n : \overline{N}(\pi) = k\}|$

Theorem: We give a formula for $|\{\pi \in \mathcal{S}_n : \overline{N}(\pi) = k\}|$; an analogous formula for $|\{\pi \in \mathcal{S}_n : N(\pi) = k\}|$ is known, [4].

Topological Entropy

If f is piecewise-monotone, the topological entropy of f of equals $\lim_{n\to\infty} \frac{1}{n} \log(|\mathrm{Allow}_n(f)|)$, [2]. We verify that

$$\lim_{n\to\infty} \frac{1}{n} \log(|\mathrm{Allow}_n(\Sigma_\sigma)|) = \log(k),$$

giving an alternative derivation of the topological entropy of Σ_σ.

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[5] S. Elizalde and K. Moore, "Patterns of Negative Shifts and Beta-Shifts". Preprint, `arXiv:1512.04479`.