Patterns in Dynamical Systems

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January 31, 2017

Brandeis University
Combinatorics Seminar

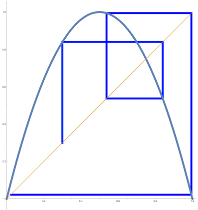
Function Iteration

Let $f : [0,1] \mapsto [0,1]$. Consider finite sequences of iterates, starting from any point x:

$$x, f(x), f(f(x)), f^3(x), \ldots, f^{n-1}(x)$$

and associate a permutation to this sequence determined by their relative order, a *pattern*.

Example: The logistic function
$f(x) = 4x(1-x)$



$x = \mathbf{.30}$
$f(x) = \mathbf{.84}$
$f(f(x)) = \mathbf{.54}$
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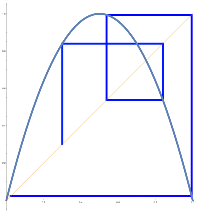

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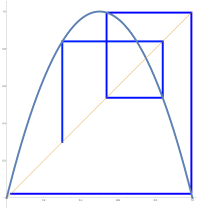

$$x = \mathbf{.30}$$
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We say **1324** *is an allowed pattern of f.*

Permutation Patterns and Dynamical Systems

Overarching Question:
What can the permutations that arise from a discrete-time dynamical system tell us about the system itself?

- A qualitative view of the short term behavior of the system.
- "Predictions" for what might occur next.
- Detecting changes in time series.

Time series analysis:

Speech signals, brain wave data (epilepsy), heart rhythms.

Forbidden Patterns

Theorem (Bandt-Keller-Pompe '02)

Every piecewise-monotone map $f : [0,1] \to [0,1]$ has forbidden patterns, i.e. patterns that never arise as iterates.

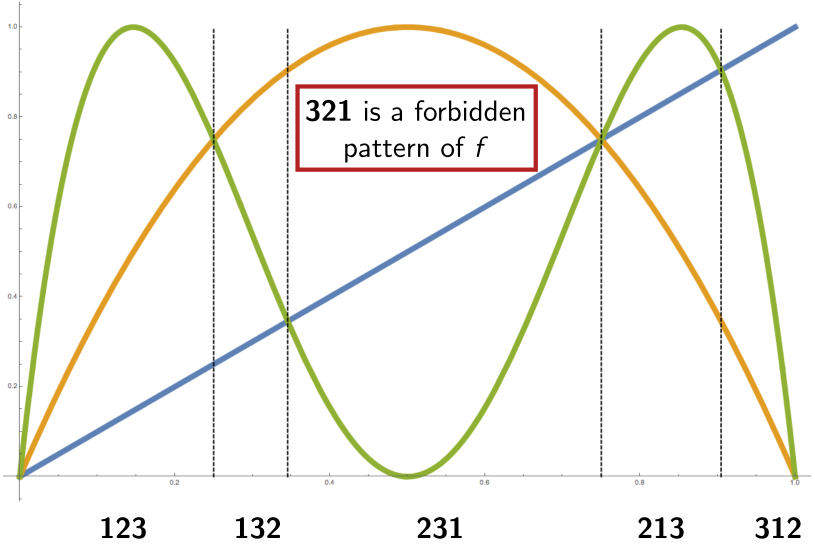
Moreover, the exponential growth rate of the number of allowed patterns is equal to the topological entropy, an important measure of complexity.

Takeaway: Patterns are a useful tool for locating determinism and estimating the complexity of a time series.

(e.g. permutation entropy)

Forbidden Patterns

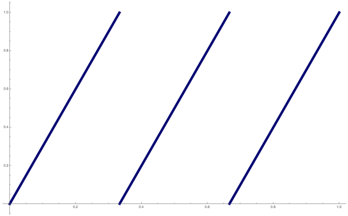
Graphs of **x**, **f(x)**, **f(f(x))** for $f(x) = 4x(1-x)$.



The K-shift

In general, determing the allowed patterns of a function is difficult.

$f(x) = \{3x\}$ (the fractional part)



$$x = \mathbf{.23}$$
$$f(x) = \mathbf{.69}$$
$$f(f(x)) = \mathbf{.07}$$
$$f(f(f(x))) = \mathbf{.21}$$

3412 *is an allowed pattern of f.*

An Illuminating Example

Let $f : [0,1] \mapsto [0,1]$ be given by $f(x) = \{10x\}$.

$x = \mathbf{.314159}$

$f(x) = \{3.14159\} = \mathbf{.14159}$

$f(f(x)) = \{1.4159\} = \mathbf{.4159}$

$f^3(x) = \{4.159\} = \mathbf{.159}$

$f^4(x) = \{1.59\} = \mathbf{.59}$

31425 is an allowed pattern of f.

An Important Idea: To study $f(x) = \{Kx\}$, write x in base K because multiplication by K is just moving the decimal place one spot to the right.

Positive Shift Map

Main idea: Take $x \in [0,1]$ and express x in base K

$$x = \frac{w_1}{K} + \frac{w_2}{K^2} + \frac{w_3}{K^3} + \dots,$$

where $w_i \in \{0, 1, \dots, K-1\}$.

Use the **lexicographical order** on these infinite K-ary words, $\mathcal{W}_K$.

Takeaway: The shift map $\Sigma : \mathcal{W}_K \mapsto \mathcal{W}_K$ given by

$$\Sigma(w_1 w_2 w_3 \dots) = w_2 w_3 w_4 \dots,$$

is now "equivalent" to the map $f(x) = \{Kx\}$.

Building a Word

Ask: Can we build a word inducing a given permutation π?

Example:

$$\pi = 523641$$

$$w = \underline{\quad 1 \quad}\ \underline{\quad 0 \quad}\ \underline{\quad 0 \quad}\ \underline{\quad \mathbf{1} \quad}\ \underline{\quad 1 \quad}\ \underline{\quad 0 \quad}\ 000\ldots$$

We call the first $n-1$ digits of w the **prefix**, $\zeta = 10011$.

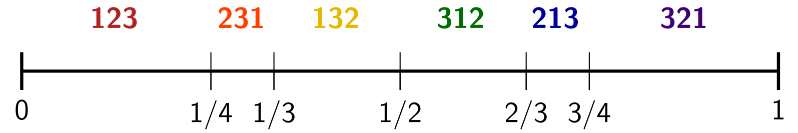
Taking shifts:

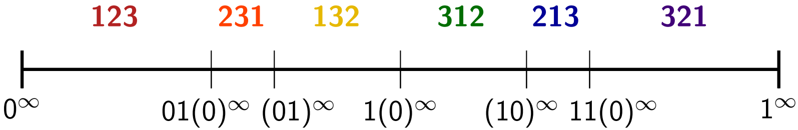
$w_{[1,\infty)} = 10011000\ldots$
$w_{[2,\infty)} = 0011000\ldots$
$w_{[3,\infty)} = 011000\ldots$
$w_{[4,\infty)} = 11000\ldots$
$w_{[5,\infty)} = 1000\ldots$
$w_{[6,\infty)} = 000\ldots$

The word w induces the permutation 523641.

The 2-shift

The intervals inducing each of permutations of length 3.





Notice how $w = 11(0)^\infty$ becomes $\frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$.

An Important Bijection

Define a bijection $\mathcal{S}_n \mapsto \mathcal{C}_n^\star$ by

$$\pi = \pi_1\pi_2\pi_3\ldots\pi_n \mapsto \hat{\pi} = (\star, \pi_2, \pi_3, \ldots, \pi_n) = \hat{\pi}_1\hat{\pi}_2\ldots\hat{\pi}_n$$

A **descent** in $\hat{\pi}$ signals the need for an additional letter.

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Example: For $\pi = 523641$ we get

$$\hat{\pi} = (\star, 2, 3, 6, 4, 1) = \star 36124.$$

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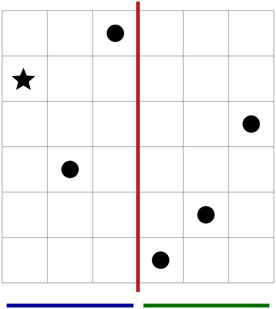

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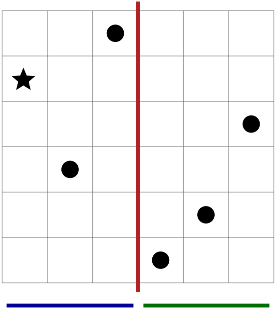


$\hat{\pi} = \star 36124$

Defining the prefix

$$\zeta = 10011$$

and

$$w = \zeta 0^\infty = 10011(0)^\infty$$

induces π.

Characterization for the K-shift

Observation: If $\pi \in \text{Allow}(\Sigma_K)$, then $\pi \in \text{Allow}(\Sigma_{K+1})$.

$$N(\pi) := \min\{K : \pi \in \text{Allow}(\Sigma_K)\}.$$

Theorem (Elizalde '09)

$$N(\pi) = 1 + des(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where $\epsilon(\hat{\pi}) = 1$ *if* $\pi_{n-1}\pi_n = 21$ *or* $\pi_{n-1}\pi_n = (n-1)n$ *and* $\epsilon(\hat{\pi}) = 0$ *otherwise.*

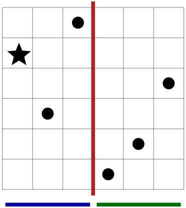
Example: For $\pi = 523641$ we have

$$\hat{\pi} = (\star, 2, 3, 6, 4, 1) = \star 36124.$$

We found the word

$$w = 10011(0)^\infty$$

and $N(\pi) = 2$.



An Exceptional Case

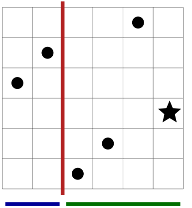
Example: For $\pi = 314256$ we get $\hat{\pi} = 45126\star$. Defining the prefix

$$\zeta = 11001.$$

Taking the sufix 1^∞ won't do because

$$w = 11001(1)^\infty$$

gives $w_{[5,\infty)} = w_{[6,\infty)} = 1^\infty$.



$\hat{\pi} = 45126\star$

We require an additional letter $\epsilon(\hat{\pi}) = 1$ and $N(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \epsilon(\hat{\pi}) = 3$.

The $(-K)$-Shift

Consider $f(x) = \{-Kx\}$ and express $x \in [0,1]$ in base $-K$

$$x = \frac{w_1}{-K} + \frac{w_2}{(-K)^2} + \frac{w_3}{(-K)^3} + \dots,$$

where $w_i \in \{0, 1, \dots, K-1\}$.

$$x = -\left(\frac{w_1+1}{-K} + \frac{w_2+1}{(-K)^2} + \frac{w_3+1}{(-K)^3} + \dots\right),$$

where $w_i \in \{0, 1, \dots, K-1\}$.

Example: Expanding .2 in base -2.

$$.2 = -\left(\frac{0+1}{-2} + \frac{1+1}{(-2)^2} + \frac{1+1}{(-2)^3} + \frac{0+1}{(-2)^4} + \dots\right)$$

$$\boxed{.2 \leftrightarrow (0110)^\infty}$$

Alternating Order: In odd positions, 0 is low and $K-1$ is high; in even positions, $K-1$ is low and 0 is high.

Characterization for $-K$-shift

Observation: If $\pi \in \mathrm{Allow}(\Sigma_{-K})$, then $\pi \in \mathrm{Allow}(\Sigma_{-(K+1)})$.

$$\boxed{\overline{N}(\pi) := \min\{K : \pi \in \mathrm{Allow}(\Sigma_{-K})\}}$$

Theorem (Elizalde and M., Charlier and Steiner)

$$\overline{N}(\pi) = 1 + asc(\hat{\pi}) + \epsilon(\hat{\pi}),$$

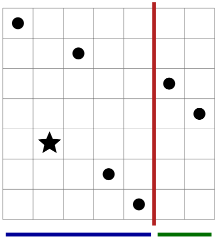
where $\epsilon(\hat{\pi}) = 1$ *if* $\pi_{n-2}\pi_{n-1}\pi_n = 2n1$ *or* $\pi_{n-2}\pi_{n-1}\pi_n = (n-1)1n$ *or* π *is "collapsed".*

Collapsed Permutations

Example:

Take $\pi = 3651742$, giving $\hat{\pi} = 7\star62154$ which defines the prefix

$$\zeta = 010010.$$



The ending must satisfy

$$w_{[4,\infty)} <_{alt} w_{[7,\infty)} <_{alt} w_{[1,\infty)}.$$

In particular,

$$(010)w_{[7,\infty)} <_{alt} w_{[7,\infty)} <_{alt} (010)(010)w_{[7,\infty)}$$

implies $w_{[7,9]} = 010$, and canceling these three letters gives

$$(010)w_{[10,\infty)} >_{alt} w_{[10,\infty)} >_{alt} (010)(010)w_{[10,\infty)}.$$

There is no word in $\mathcal{W}_2$ with $<_{alt}$ inducing π.
We require an additional letter; $\epsilon(\hat{\pi}) = 1$ and $\overline{N}(\pi) = 3$.

Enumerations

Theorem (Elizalde '09)

Let $b(n,k) = |Allow_n(\Sigma_K) \setminus Allow_n(\Sigma_{(K-1)})|$. *Then*

$$\sum_{k=2}^{n} b(n,k)x^k = (1-x)^n \sum_{k\geq 2} p(n,k)x^k,$$

where

$$p(n,k) = (k-2)k^{n-2} + \sum_{i=1}^{n-1} k^{n-i-1}\psi_k(i),$$

and $\psi_k(i) = \sum_{d|i} \mu(\frac{i}{d})k^t$ *is the number of primitive words of length* i *on* k *letters.*

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where

$$p(n,k) = (k^2-2)k^{n-3} + \sum_{i=1}^{n-1} k^{n-i-1}\psi_k(i)$$

$$-2\sum_{j=1}^{k-1} j^{n-3} - 2\sum_{\substack{c=1\\ c\ odd}}^{n-1}\sum_{j=1}^{k-1}\frac{c-1}{c}\binom{c+k-j-1}{k-j} j^{n-2c-1}\psi_j(c),$$

and $\psi_k(i) = \sum_{d|i}\mu(\frac{i}{d})k^t$ *is the number of primitive words of length* i *on* k *letters.*

Enumerations

Allowed patterns for the K-shift.

n \ K	2	3	4	5	6	7
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

Allowed patterns for the $-K$-shift.

n \ $-K$	2	3	4	5	6	7
3	6					
4	20	4				
5	54	62	4			
6	140	408	168	4		
7	336	2084	2196	412	4	
8	800	9152	19556	9804	972	4

Smallest Forbidden Patterns

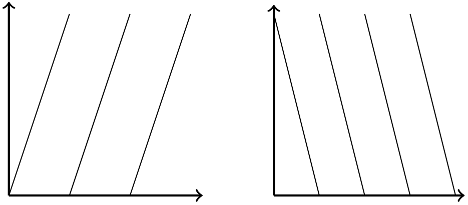
The smallest forbidden patterns of the 4-shift are

$$615243, 324156, 342516, 162534, 453621, 435261.$$

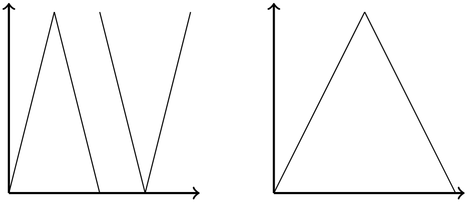
The smallest forbidden patterns of the -4-shift are

$$123456, 654321, 123465, 654312.$$

Signed-Shifts



The graphs of M_σ for $\sigma = +^3$, $\sigma = -^4$.



The graphs of M_σ for $\sigma = + - - +$, and $\sigma = + -$, respectively.

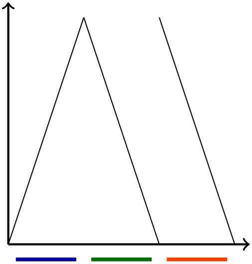
Words and Orders

Example: Consider M_σ for $\sigma = + - -$.

The set of words is $\mathcal{W}_3 = \{0, 1, 2\}^\infty$.

The corresponding order, $<_\sigma$, is given by:
$v_1 v_2 v_3 \ldots <_\sigma w_1 w_2 w_3 \ldots$ if

- $v_1 < w_1$
- $\mathbf{v_1 = w_1 = 0}$ and $v_2 v_3 \ldots <_\sigma w_2 w_3 \ldots$
- $\mathbf{v_1 = w_1 = 1}$ and $v_2 v_3 \ldots >_\sigma w_2 w_3 \ldots$
- $\mathbf{v_1 = w_1 = 2}$ and $v_2 v_3 \ldots >_\sigma w_2 w_3 \ldots$



Segmentations

Recall the bijection $\mathcal{S}_n \mapsto \mathcal{C}_n^\star$ by

$$\pi = \pi_1\pi_2\pi_3\ldots\pi_n \mapsto (\star, \pi_2, \pi_3, \ldots, \pi_n) \mapsto \hat{\pi} = \hat{\pi}_1\hat{\pi}_2\ldots\hat{\pi}_n.$$

Example: Take $\sigma = +-$.
For $\pi = 58123647$ we get

$$\hat{\pi} = (\star, 8, 1, 2, 3, 6, 4, 7) = 236784\star 1.$$

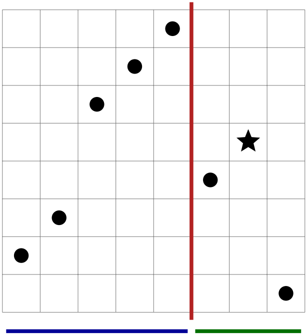
Defining the prefix

$$\zeta = 0100010$$

and

$$w = \zeta(100)^\infty = 0100010(100)^\infty$$

induces π. Therefore $\pi \in \text{Allow}(M_{+-})$.



Signed-Shift Characterization

Theorem (Archer, Elizalde, M.)

A permutation, π, is realized by the signed-shift M_σ if and only if * $\hat{\pi}$ *is the same shape as* M_σ.

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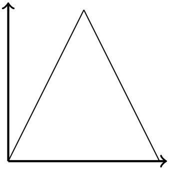
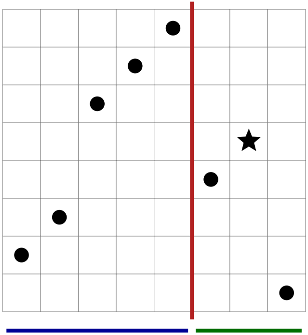
Example:

Consider $\sigma = +-$ (the tent map).



The permutation $\pi = 58123647$ is realized by M_σ because $\hat{\pi} = 23674\star1$ is unimodal.



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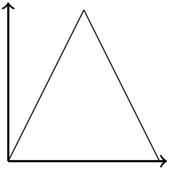

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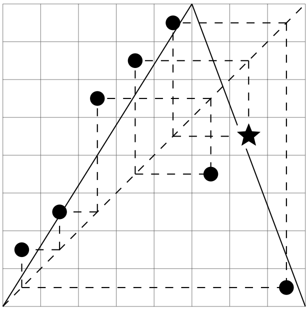

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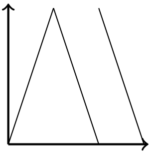
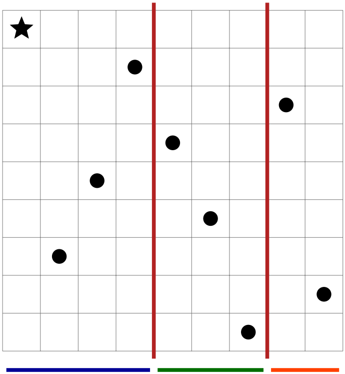


The permutation $\pi = 923564871$ is realized by M_σ because $\hat{\pi} = \star 35864172$ is the same shape as M_σ.



Signed-Shift Characterization

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Under the Rug:

Collapsed permutations and $\pi_{n-1}\pi_n = (n-1)n$ when $\sigma_k = +$, etc.

The Reason Why:

The cycle diagram of $\hat{\pi}$ is the cobweb diagram for M_σ starting at a point that induces π.

Beta-shifts and Negative Beta-shifts

Consider the classes of functions,

$$T_\beta(x) = \{\beta x\} \text{ and } T_{-\beta}(x) = \{-\beta x\},$$

where $\beta > 1$ is a real number.

Key Idea: As β is increased, patterns increase by containment.

Ask: What is the infimum β such that $\pi \in \text{Allow}(T_\beta)$?
(resp. $\pi \in \text{Allow}(T_{-\beta})$)

Answer: The infimum can be expressed as the largest root of a polynomial with integer coefficients.
(Elizalde '09) and (Charlier and Steiner, Elizalde and M.)

Beta-shifts and Negative Beta-shifts

The permutation $\pi = 25341$ is realized by

- The map $T_\beta(x) = \{\beta x\}$ for all $\beta > 2.65897$; the infimum is the largest root of

$$\beta^3 - 2\beta^2 - \beta - 2.$$

- The map $T_{-\beta}(x) = \{-\beta x\}$ for all $\beta > 1.7549$; the infimum is the largest root of

$$\beta^3 - 2\beta^2 + \beta - 1.$$

Periodic Points

Permutations occuring at periodic points follow analogous ideas.

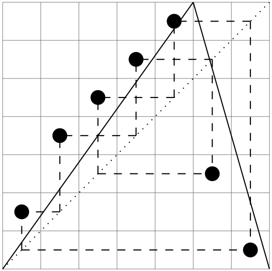
Similar to questions about the forcing relations arising from Sarkovskii's Theorem.

Relating cycle structure and one-line notation:

Theorem (Archer and Elizalde '13)

$$|\mathcal{C}_n(213, 312)| = \frac{1}{2n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{n/d},$$

these are n-cycles whose one-line permutation can be drawn on $\sigma = +-$.



Further Questions

- Can we understand allowed patterns of other functions? Progress on symmetric tent map. (Acher and LaLonde '16)
- Can we use allowed patterns to describe the dynamics present in time-series?
- Can we understand patterns in different contexts, such as random walks? (Martinez '15)
- Can we determine the probability of certain patterns occuring?

References

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